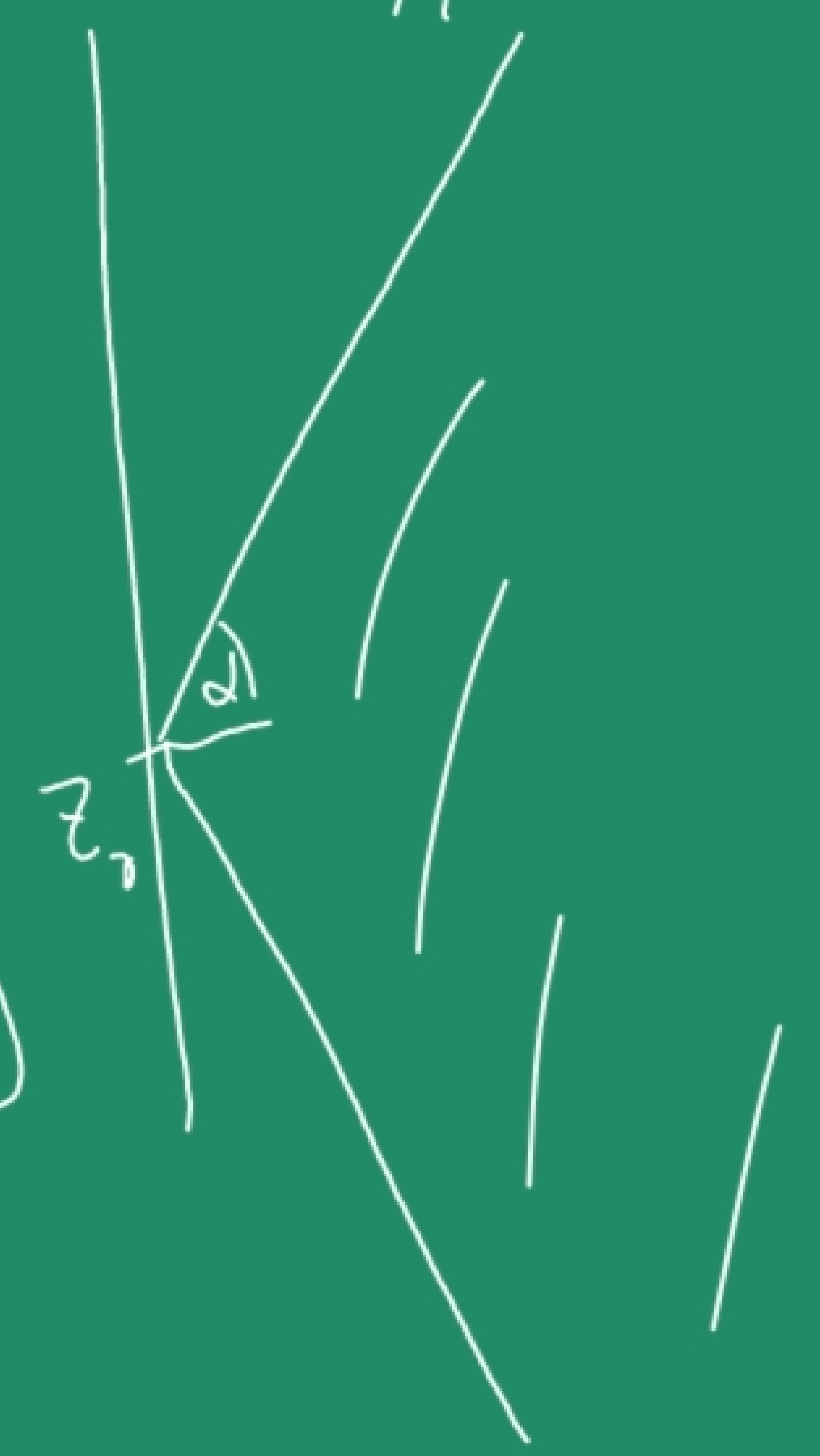
Áll.: Tfh. $f(z) = \sum_{n=1}^{\infty} \frac{a_n}{n^z}$ konvergál $z = z_0 \in \mathbb{C}$-ra.

$\Rightarrow$ egyenletesen konvergál $\mathrm{Re}(z) \geq \mathrm{Re}(z_0)$, $|\arg(z - z_0)| \leq \alpha$

tartományon $\forall\ \alpha < \frac{\pi}{2}$-re.



Biz.: $\left[f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \right.$

$0 \leq \lambda_1 < \lambda_2 < \dots \quad \lambda_n \to +\infty \Big]$

nálunk: $\lambda_n = \log n$

$z_0 = 0$ feltehető. $b_n = \frac{a_n}{n^{z_0}}$ $\quad z \to z - z_0$ $\quad z$

$\frac{a_n}{n^z} = b_n \cdot \frac{1}{n^{z-z_0}}$

$z_0 = 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ konvergens.

$Re(z) \geq 0$

$\forall \varepsilon > 0 \quad \exists N > 0 \quad m, m' \geq N$-re

$$\Big|\underbrace{\sum_{n=m}^{m'} a_n}_{A_{m,m'}}\Big| < \varepsilon.$$

$A_{m,m-1} = 0$

$A_{m,m} = a_m$

$$S_{m,m'} = \sum_{n=m}^{m'} \frac{a_n}{n^z} = \sum_{n=m}^{m'} \frac{A_{m,n} - A_{m,n-1}}{n^z} = \sum_{n=m}^{m'-1} A_{m,n}\left(\frac{1}{n^z} - \frac{1}{(n+1)^z}\right) + \underbrace{A_{m,m'}}_{<\varepsilon} \cdot \frac{1}{m'^z}$$

$a_n = A_{m,n} - A_{m,n-1}$

Abel-átrendezés

Lemma $0 < \alpha < \beta \in \mathbb{R}$ $\quad z = x + iy \quad x > 0.$

$$\left| e^{-\alpha z} - e^{-\beta z} \right| \leq \frac{|z|}{x} \left(e^{-\alpha x} - e^{-\beta x} \right)$$

Biz.:

$$e^{-\alpha z} - e^{-\beta z} = z \int_{\alpha}^{\beta} e^{-tz} \, dt \qquad \left(e^{-tz} \right)' = -z e^{-tz}$$

$$\frac{e^{-tz}}{z}$$

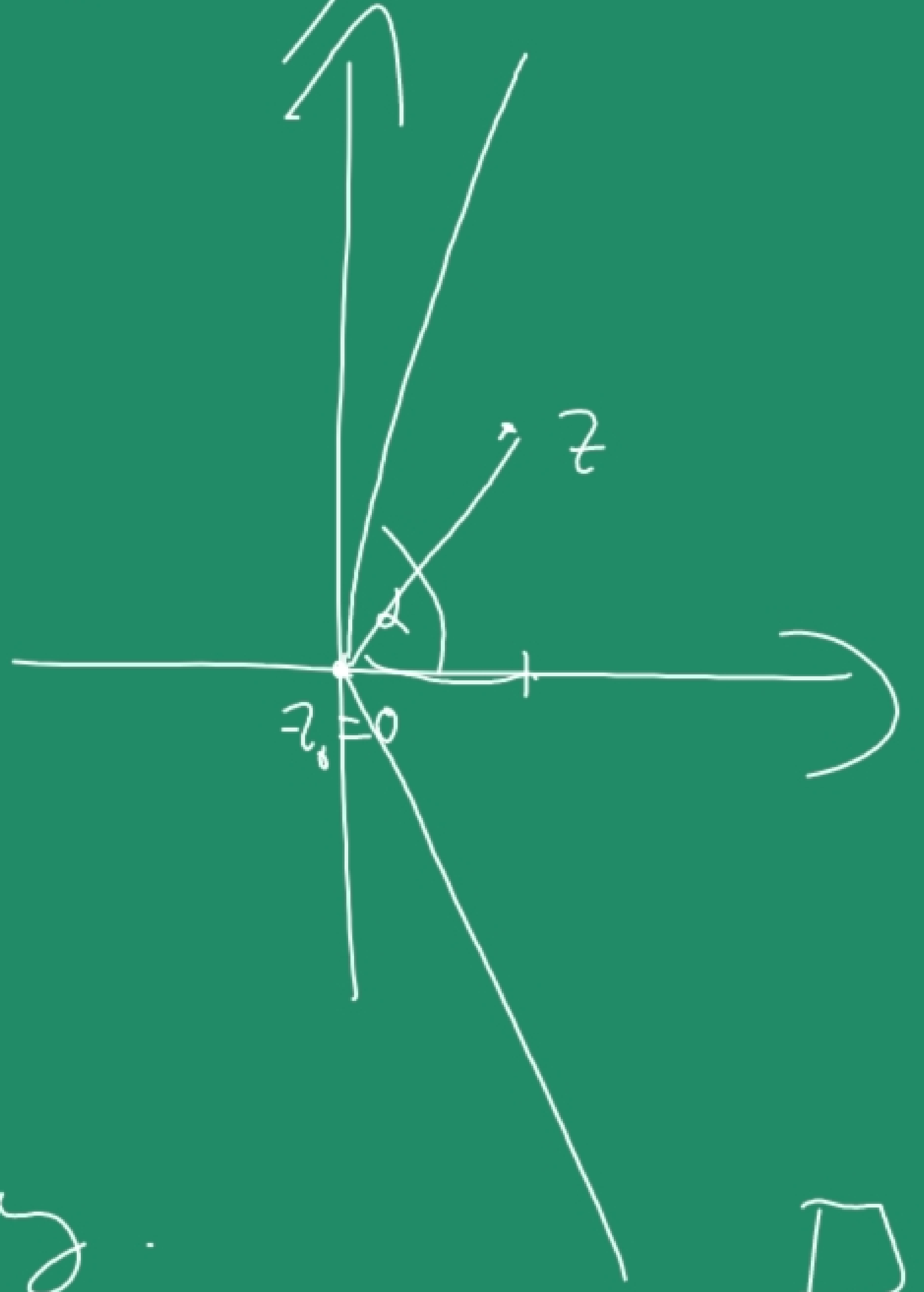
$$\left| e^{-tz} \right| = e^{-tx} \qquad \left| e^{-tiy} \right| = 1$$

$$\left| e^{-\alpha z} - e^{-\beta z} \right| \leq |z| \cdot \int_{\alpha}^{\beta} e^{-tx} \, dt = \frac{|z|}{x} \left(e^{-\alpha x} - e^{-\beta x} \right) \quad \square$$

$$\Rightarrow \quad S_{m,m'} \leq \varepsilon \left(1 + \frac{|z|}{x} \sum_{n=m}^{m'-1} \left(\frac{1}{n^x} - \frac{1}{(n+1)^x} \right) \right) \overset{*}{\leq}$$

$$\frac{|z|}{x} \leq \frac{1}{\cos\alpha}$$

$$\cos\alpha \leq \frac{x}{|z|}$$



$$\overset{*}{\leq} \varepsilon \left(1 + \frac{1}{\cos\alpha} \underbrace{\left(\frac{1}{m^x} - \frac{1}{m'^x} \right)}_{\leq 1} \right)$$

$\Rightarrow$ egyenletesen Cauchy. $\square$

Köv.: $f(z)=\sum_{n=1}^{\infty} \frac{a_n}{n^z}$ komplex differenciálható $\mathrm{Re}(z)>\mathrm{Re}(z_0)$

(és folytonos).

Biz.: Cauchy-formula: $z_0 \in U \subset \mathbb{C}$ nyílt

$f: U \to \mathbb{C}$ diff.-ható (komplex) holomorf

$$f'(z_0)=\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^2}\,dz$$

$$f(z_0)=\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,dz$$

$f=\lim f_n$ egyenletesen U-n, f_n diffható

$$f_n'(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f_n(z)}{(z-z_0)^2}\,dz$$

$f_n'(z_0)$ sorozat Cauchy.

$$f'(z_0) := \lim_{n\to\infty} f_n'(z_0).$$

$$f_N(z) = \sum_{n=1}^{N} \frac{a_n}{n^z}$$ komplex diffható.

$f_N \to f$ egyenletesen

$\square$

Köv2 $f(z) \equiv 0$ azonosan $0 \Rightarrow a_n = 0 \;\forall n \geq 1$-re.

Biz.: n szerinti indukció.

$$a_1 = \lim_{z \to +\infty} f(z)$$

$$a_1 + \underbrace{\frac{a_2}{2^z} + \frac{a_3}{3^z} + \dots}_{\to 0}$$

$$a_1 = \dots = a_{n-1} = 0$$

$$\lim_{z \to +\infty} n^z f(z) = a_n = 0 \qquad \square$$

Köv3 $|a_n|$ korlátos $\Rightarrow \sum_{n=1}^{\infty} \frac{a_n}{n^z}$ konvergens $Re(z) > 1$

Köv4 $A_{m,m'}$ korlátos $\Rightarrow$ $z_0 = 1+\varepsilon$ -ra abs. konv.

$$f(z) = \sum \frac{a_n}{n^z} \text{ konvergens } Re(z) > 0$$

Abel-átrendezés. ra.

$f : \mathbb{N} \longrightarrow \mathbb{C}$ multiplikatív $\quad f(ab) = f(a)f(b)$ ha $(a,b) = 1$

$$L(f,s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

f korlátos

konvergens $\mathrm{Re}(s) > 1$-re

<u>Lemma</u> $\mathrm{Re}(s) > 1 \Rightarrow L(f,s) = \prod_{p \text{ prím}} \left(1 + \frac{f(p)}{p^s} + \dots + \frac{f(p^k)}{p^{ks}} + \dots\right)$

<u>Biz.:</u> $n = p_1^{\alpha_1} \cdot \ldots \cdot p_r^{\alpha_r}$ kiszorozva a jobb oldalon

$f(n) = f(p_1^{\alpha_1}) \cdot \ldots \cdot f(p_r^{\alpha_r})$ az $\frac{1}{n^s}$ eh.-ja

$\square$

Tfh. $f(ab) = f(a)f(b)$ $\forall a, b \in \mathbb{N}$-re

(teljesen mult.), f korlátos.

$\Rightarrow L(f,s) = \prod_{p \text{ prím}} \frac{1}{1 - \frac{f(p)}{p^s}}$, mert

Euler-szorzat alak.

$$\frac{1}{1 - \frac{f(p)}{p^s}} = 1 + \frac{f(p)}{p^s} + \frac{f(p)^2}{p^{2s}} + \dots$$

$$f(p)^2 = f(p^2).$$

<u>Riemann-zeta fv.:</u> $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}$

$(\mathrm{Re}(s) > 1).$

$\chi : \mathbb{Z} \longrightarrow \mathbb{C}^\times$ mod m mult. karakter.

$\chi \quad \underbrace{\left(\mathbb{Z}/(m)\right)^\times}_{\varphi(m)\text{-elemű}} \longrightarrow \mathbb{C}^\times$ (csop. hom.

pl.: $m = 4$, $\chi(k) = \begin{cases} 1 & k \equiv 1\ (4) \\ -1 & k \equiv 3\ (4) \end{cases}$

$m = p$ prím

$\chi(a) := \begin{cases} 0 & (a,m) \neq 1 \\ \chi(a \bmod m) & (a,m) = 1. \end{cases}$

$\chi(g) = \zeta_{p-1} \rightsquigarrow$ prim. $(p-1)$-edik egység.

g prim. gyök mod p

$\chi(g^k) = \zeta_{p-1}^k$.

$L(\chi, s)$ konvergens $Re(s) > 1$-re $\forall \chi$ esetén

és ha $\chi \neq \mathbb{1}_m$, akkor $Re(s) > 0$-ra is.

<u>Áll.:</u> ζ komplex diffható $Re(s) > 1$-re és

$\zeta(s) = \frac{1}{s-1} + \phi(s)$, ahol $\phi(s)$ kiterjed $Re(s) > 0$-ra

is komplex diffhatóan. (Valójában egész $\mathbb{C}$-re is kiterjed)

Riz.:

$$\zeta(s) - \frac{1}{s-1}$$

$$\frac{1}{s-1} = \int_1^\infty t^{-s}\,dt = \sum_{n=1}^{\infty} \int_n^{n+1} t^{-s}\,dt$$

$$\left(t^{1-s}\right)' = (1-s)\,t^{-s}$$

konvergens $Re(s) > 0$

$$\phi(s) = \zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} \left(\frac{1}{n^s} - \int_n^{n+1} t^{-s}\,dt \right) = \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{t^s} \right) dt$$

$$\left| \frac{1}{n^s} - \frac{1}{t^s} \right| \leq \frac{|s|}{Re(s)} \left(\frac{1}{n^{Re(s)}} - \frac{1}{t^{Re(s)}} \right)$$

$$\leq \left(\frac{1}{n^{Re(s)}} - \frac{1}{(n+1)^{Re(s)}} \right) \frac{|s|}{Re(s)}$$

Köv.:

$$s \to 1 \Rightarrow \sum_{p \text{ prím}} \frac{1}{p^s} \sim \log \frac{1}{s-1}$$ divergens.

$$\sum_{p,\ 1 \leq k \geq 2} \frac{1}{p^{ks}}$$ korlátos.

Biz.:

$$\log \zeta(s) = \log \prod_p \frac{1}{1-p^{-s}} = \sum_p \log\left(\frac{1}{1-\frac{1}{p^s}}\right)$$

$$= \sum_{\substack{p,\ k \geq 1 \\ \text{prím}}} \frac{1}{k \cdot p^{k \cdot s}}$$

$$\log \frac{1}{1-x} = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \frac{x^n}{n}(-1)^{n+1}$$

$|x| < 1$.

$$\sum_{p,\, q \geq 2} \frac{1}{q \cdot p^{qs}} \leq \sum_{p,\, q \geq 2} \frac{1}{p^{qs}} \leq \sum_{p} \frac{1}{p^s(p^s - 1)} \leq \sum_{p} \frac{1}{p(p-1)}$$

$$\leq \sum_{n>1} \frac{1}{n(n-1)} = 1$$

konv.

$\mathrm{Re}(s) > 1$

$$\frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \dots = \frac{1}{p^s(p^s - 1)}$$

$$\zeta(s) = \frac{1}{s-1} + \phi(s)$$

$\mathrm{Re}(s) > 1$

$$\log(\zeta(s)) = \log\left(\frac{1}{s-1} + \phi(s)\right) = \log\frac{1}{s-1} + \text{korlátos}$$

$$\| $$

$$\sum_{p \text{ prím}} \frac{1}{p^s} + \text{korlátos}$$

$m > 1$ egész $\quad p \nmid m$ prím.

$f(p) := $ a rendje mod m: $\quad p^{f(p)} \equiv 1 \quad (m)$

$f(p)$ minimális.

$$g(p) := \frac{\varphi(m)}{f(p)} \qquad f(p) \mid \varphi(m).$$

Lemma

$$\prod_{\chi} \left(1 - \chi(p) T\right) = \left(1 - T^{f(p)}\right)^{g(p)}$$

$\chi : \left(\mathbb{Z}/(m)\right)^{\times} \to \mathbb{C}^{\times}$

karakter

<u>Biz.:</u> $\langle p \rangle \leq \left(\mathbb{Z}/(m)\right)^{\times}$ $f(p)$-rendű ciklikus.

$\chi(p)$ $f(p)$-edik egységgyök.

$$\prod_{\chi}\left(1-\chi(p)T\right) = \Big(\underbrace{\prod_{\zeta}(1-\zeta T)}_{\substack{\zeta \\ f(p)\text{-edik egys. gyök.}}}\Big)^{g(p)} = \left(1-T^{f(p)}\right)^{g(p)}$$

$\square$

Köv.: $\zeta_m(s) := \prod_{\chi : (\mathbb{Z}/(m))^\times \to \mathbb{C}^\times} L(\chi, s) = \prod_{p \nmid m} \frac{1}{\left(1 - \frac{1}{p^{f(p)s}}\right)^{g(p)}}$

$T = \frac{1}{p^s}$ - re az előző lemma.

$\Rightarrow$ nem terjed ki $s = 1$-re.

$L(\mathbb{1}_m, s) = \prod_{p \nmid m} \frac{1}{1 - p^{-s}}$

$\Rightarrow L(\chi, 1) \neq 0 \quad \chi \neq \mathbb{1}_m$.

Tétel (Dirichlet) $(a,m)=1$ $\quad P_a = \left\{ p \text{ prím} \;\middle|\; p \equiv a \ (m) \right\}$.

$$\sum_{p\in P_a} \frac{1}{p^s} \Big/ \log\frac{1}{s-1} \longrightarrow \frac{1}{\varphi(m)} \quad \text{ha } s \to 1$$

Speciálisan P_a végtelen.

Biz.: $g_a(s) = \sum_{p\in P_a} \frac{1}{p^s} = \frac{1}{\varphi(m)} \sum_{\chi:(\mathbb{Z}/(m))^\times \to \mathbb{C}^\times} \chi(a)^{-1} f_\chi(s)$

$$f_\chi(s) = \sum_{p\nmid m} \frac{\chi(p)}{p^s}.$$

$$\frac{1}{\varphi(m)} \sum_\chi \frac{\chi(p)}{\chi(a)} = \begin{cases} 1 & p \equiv a \\ 0 & p \not\equiv a \end{cases}$$

$L(\chi,1)\neq 0$

$$\underbrace{\log L(\chi,s)}_{\text{φραγμένος}} = \sum_p \log \frac{1}{1-\frac{\chi(p)}{p^s}} = \sum_{n,p} \frac{\chi(p)^n}{n p^{ns}} =$$

$s \to 1$.

$$= f_\chi(s) + \underbrace{F_\chi(s)}_{\text{φραγμένος}}.$$

$\chi \neq \mathbb{1}_m$ $\quad f_\chi(s)$ φραγμένος)

$\chi = \mathbb{1}_m$ $\quad f_{\mathbb{1}_m}(s) \sim \log \frac{1}{1-s}$

$$g_a(s) \sim \frac{1}{\varphi(m)} \log \frac{1}{1-s} \qquad \square$$