Tétel Az e transzcendens.

Biz.: Tfh. $g(e)=0$, ahol $g(x)=b_0+b_1x+\dots+b_rx^r\in\mathbb{Z}[x]$.

$$I_f(t)=\int_0^t e^{t-u}f(u)\,du$$

Legyen $p>r$, $|b_0|$ p-nél kisebb prím

$$f(x)=x^{p-1}(x-1)^p(x-2)^p\cdot\dots\cdot(x-r)^p$$

$$\deg f=rp+p-1=r(p+1)-1=:n$$

$$J:=b_0I_f(0)+b_1I_f(1)+\dots+b_rI_f(r)$$ mert $g(e)=0$

Lemma 1 $\Rightarrow$ $$J=\sum_{i=0}^{r}b_i\sum_{j=0}^{n}\left(e^i f^{(j)}(0)-f^{(j)}(i)\right)=$$

$I_f^0(x)$

$$= - \sum_{\ell=0}^{r} \sum_{j=0}^{h} b_\ell f^{(j)}(\ell)$$

$f^{(j)}(i) = 0$, ha $0 \le j \le p-1$ e's $i = 1, \dots, r$

$f^{(j)}(0) = 0$ ha $0 \le j \le p-2$

$$J = - \sum_{j=p-1}^{m} b_0 f^{(j)}(0) - \sum_{\ell=1}^{r} \sum_{j=p}^{h} b_\ell f^{(j)}(\ell)$$

$$(x^m)^{(j)} = \underbrace{m(m-1)\cdots(m-j+1)}_{j!} x^{m-j}$$

$\Rightarrow (p-1)! \mid J$, de $p \nmid c_0 f^{(p-1)}(0)$

miatt $p \nmid J$.

$p \nmid f^{(p-1)}(0) = (p-1)! (\pm 1)^{rp} (r!)^p$

$f(x) = x^{p-1}(x-1)^p \cdots (x-r)^p$

$\Rightarrow (p-1)! \leq |J|$.

Felső becslés J-re:

$$|J| \leq \sum_{j=0}^{r} |c_j| \left| T_f(j) \right| \overset{\text{L7}}{\leq} \sum_{j=0}^{r} |c_j| \cdot j e^j \widetilde{f}(j) \leq$$

$$\leq c \cdot (2r)^{(r+1)p}$$

Ly, mert $p \gg 0$ esetén $(p-1)! > c \cdot A^p$

A, c konstans]

Tétel π transzcendens.

Biz.: Tfh. algebrai $\Rightarrow$ πi is algebrai.

$\exists\, g(x) \in \mathbb{Z}[x]$, $g(x) = bx^r + \dots$, $b \in \mathbb{Z}$, $b := g(0)$

$g(\pi i) = 0$. $\deg g = r$, jöh.-ja

$\alpha_1 = \pi i$, $\alpha_2, \dots, \alpha_r \in \mathbb{C}$ a g gyökei

$$\prod_{j=1}^{r} (1 + e^{\alpha_j}) = 0,$$ mert $e^{\pi i} = -1$.

2^r de rag, e^{ϕ} alarinar, ahol $\phi = \varepsilon_1 \alpha_1 + \ldots + \varepsilon_r \alpha_r$ alaru, $\varepsilon_j \in \{0;1\}$.

$\phi_1, \ldots, \phi_n :=$ nem nulla tagok a ϕ-k közül

$q := 2^r - n$ db 0.

$$q + e^{\phi_1} + \ldots + e^{\phi_n} = 0.$$

$p \gg 0$ prim.

$$f(x) = e^{np} x^{p-1} (x - \phi_1)^p \cdot \ldots \cdot (x - \phi_n)^p$$

Áll.: $f(x) \in \mathbb{Z}[x]$.

Biz.: $b\alpha_j \in \overline{\mathbb{Z}}$ $\quad (j = 1, \dots, r)$

$$b\alpha_j^r + b_1\alpha_j^{r-1} + \dots + b_r = 0$$

b^{r-1} /

$$(b\alpha_j)^r + b_1(b\alpha_j)^{r-1} + \dots + b^{r-1}b_r = 0$$

$$f(x) = b^{n-1}x^{s-1}(x - \beta_1)^p \cdot \dots \cdot (x - \beta_n)^p \in \overline{\mathbb{Z}}[x].$$

$$b\beta_j \in \overline{\mathbb{Z}}$$

Másrészt a $G := Gal(K_g/\mathbb{Q})$, ahol

K_g : g felb. teste $\Rightarrow$ G a $\phi_1, \dots, \phi_n$ számokat egymás közt permutálja

$\Rightarrow$ $x^q(x-\phi_1)\dots(x-\phi_n)$ G-invariáns.

$\Rightarrow$ eh.-i G-inv.-ak $\Rightarrow$ (Galois főtétel) $\in \mathbb{Q}$.

$\mathbb{Q} \cap \overline{\mathbb{Z}} = \mathbb{Z}$ $\Rightarrow$ $f(x) \in \mathbb{Z}[x]$.

□

Le gyen $J := \overline{I}_g(\phi_1) + \dots + \overline{I}_g(\phi_n)$.

$\deg J = m = (n+1)p - 1$

lemma 1 $\Rightarrow J = \sum_{\ell=1}^{n} \sum_{j=0}^{m} \left(e^{\phi_\ell} g^{(j)}(0) - g^{(j)}(\phi_\ell) \right) =$

$$= -q \sum_{j=0}^{m} g^{(j)}(0) +$$

$$- \sum_{\ell=1}^{n} \sum_{j=0}^{m} g^{(j)}(\phi_\ell).$$

$\in \mathbb{Q}$

$$\sum_{\ell=1}^{n} e^{\phi_\ell} = -q$$

$f^{(j)}(\phi_k) \in \overline{\mathbb{Z}}$, mert $f(x) = b^{np} x^{p-1}(x-\phi_1)^p \cdots (x-\phi_n)^p$

mert $b\phi_j \in \overline{\mathbb{Z}}$ $\forall j = 1, \ldots, n-r$.

$j \leq p-1 \Rightarrow f^{(j)}(\phi_k) = 0$.

$\Rightarrow$

$j \leq p-2$ $f^{(j)}(0) = 0$

$f^{(p-1)}(0) \neq 0$ (ugyanúgy)

$p > |(b\phi_1) \cdots (b\phi_n)|^p$ elég.

$\Rightarrow J \in \mathbb{Q} \cap \overline{\mathbb{Z}} = \mathbb{Z}$ és $(p-1)! \mid J$, de

$p \nmid J \Rightarrow |J| \geq (p-1)!$

$$(p-1)! \leq |J| \leq \sum_{i=1}^{n} |\phi_i| e^{|\phi_i|} \tilde{f}(|\phi_i|) \leq c \cdot A^p$$

$$f(x) = a^{np} x^{p-1} (x-\phi_1)^p \dots (x-\phi_n)^p$$

$$\tilde{f}(x) = a^{np} x^{p-1} (x+|\phi_1|)^p \dots (x+|\phi_n|)^p$$

$$\left(a^n 2^{n(n-1)} (|\phi_1| + \dots + |\phi_n|)^{n(n-1)} \right)^p \cdot (x+|\phi_1|)^p \dots (x+|\phi_n|)^p$$

A

Ha p

p-adikus számok

$\frac{a}{b} \in \mathbb{Q}$ $\quad (a,b)=1$ $\quad p^{\xi} \frac{a_1}{b_1} = \frac{a}{b}$, $p \nmid a_1, b_1$. $\quad \xi \in \mathbb{Z}$

Def.: $\left|\frac{a}{b}\right|_p := p^{-\xi}$. (p-adikus abszolútérték).

$|0|_p := 0$.

Áll.: 1) $|x|_p = 0 \iff x = 0$

2) $|xy|_p = |x|_p |y|_p$

3) $|x+y|_p \leq |x|_p + |y|_p$, sőt 3') $|x+y|_p \leq \max(|x|_p, |y|_p)$

ultrametrikus egyenlőtlenség.

Biz.: a) $|x|_p = 0 \Leftrightarrow x = 0$ ✓ $\qquad p^{-q} > 0$

b) $|xy|_p = p^{-q-\ell} = |x|_p \cdot |y|_p$

$x = p^q x_1 \qquad xy = p^{q+\ell} x_1 y_1$

$y = p^\ell y_1$

c) $x = p^q x_1, \quad y = p^\ell y_1 \qquad v_p(x_1) = v_p(y_1) = 0 \quad (v_p(a) = q,$ ha $a = p^q \cdot \frac{b}{c}$ $p \nmid b, c.)$

$q \le \ell$

$x + y = p^q \left(x_1 + p^{\ell - q} y_1 \right) \Rightarrow |x+y|_p \le p^{-q}$

□

Kicsi? p^N (N nagy)

Cauchy-sorozat: $(a_n)_{n\geq 1} \subset \mathbb{Q}$ Cauchy, ha

$\forall \varepsilon > 0 \ \exists N$ úgy, hogy $n, m \geq N$ $|a_n - a_m|_p < \varepsilon$.

elég: $\forall n \geq N$ $|a_{n+1} - a_n|_p < \varepsilon$.

$m > n$ $|a_m - a_n|_p \leq \max(|a_m - a_{m-1}|_p, \ldots, |a_{n+1} - a_n|_p)$

<u>köv.:</u> $|a_n|_p \to 0 \Rightarrow \sum_{n=0} a_n$ konvergens

$\mathbb{Q}_p := \mathbb{Q}$ teljesítése $|\cdot|_p$-re nézve = Cauchy-sorozatok ekvivalenciaosztályai.

$(a_n)_{n\geq 1} \sim (b_n)_{n\geq 1}$
$\Leftrightarrow |a_n - b_n|_p \to 0.$

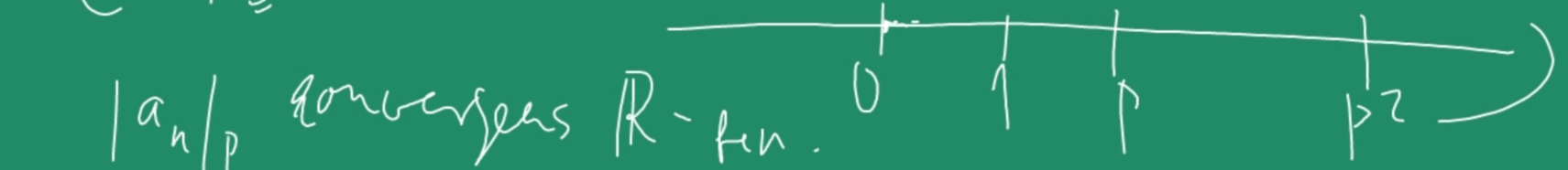
$|\cdot|_p : \mathbb{Q}_p \to \mathbb{R}^{\geq 0}$

(sőt $\{0\} \cup p^{\mathbb{Z}}$)

$|\cdot|_p : \mathbb{Q} \to \mathbb{R}^{\geq 0}$ értékkészlete: $\{0\} \cup p^{\mathbb{Z}}$.

$(a_n)_{n\geq 1}$ Cauchy $|\cdot|_p$

$|a_n|_p$ konvergens $\mathbb{R}$-ben.



$\Rightarrow$ $|\cdot|_p : \mathbb{Q}_p$ ultrametric valuation $\{0\} \cup p^{\mathbb{Z}}$.

$0 \neq x \in \mathbb{Q}_p$ $\quad |x|_p \neq 0$ $\quad x \in \mathbb{Z}$

$|x|_p = p^{-r}$

$|p^r x|_p = 1$.

$|p^r| = p^{-r}$.

$\left((a_n)_{n \geq 1} \right) = p^r x$ $\quad a_n \longrightarrow p^r x$ $\quad |\cdot|_p - \text{Cauchy}$.

$n \gg 0 \Rightarrow |a_n|_p = 1$. $\quad a_n = \frac{b_n}{c_n} \in \mathbb{Q}$

$p \nmid b_n, c_n \in \mathbb{Z}$

$$p^g x = x_0 + p x_1 + p^2 x_2 + \dots + p^r x_r + \dots$$

$$x_r \in \{0, 1, \dots, p-1\}.$$

$$x = p^{-g} x_0 + p^{-g+1} x_1 + \dots$$

$$-1 = (p-1) + p((p-1) + \dots = (p-1)(1 + p + \dots + p^r + \dots)$$

$$-1 = p - 1 + py \qquad = \frac{p-1}{1-p} = -1$$

$$y = -1$$

Hasse-Minkowski-tétel: $f(x_1, \dots, x_n)$ nem-elfajuló n-változós homogén 2-fokú pol. $\in \mathbb{Q}[x_1, \dots, x_n]$. $f(x_1, \dots, x_n) = 0$-nak $\exists$ nemtriviális gyöke $\mathbb{Q}$-ban $\iff$ $\exists$ nemtriviális gyök $\mathbb{R}$-ben és $\forall p$ prímre $\mathbb{Q}_p$-ben.

$$\mathbb{Z}_p := \{ x \in \mathbb{Q}_p \mid |x|_p \leq 1 \} \to p\text{-adikus egészek}.$$

$$\| \quad x_0 + x_1 p + \dots$$

$$x_0 \in \{0, \dots, p-1\}$$

$\mathbb{Q}_p = \mathbb{Z}_p[p^{-1}]$. $\quad 1 + p + 2p^2 +$

$\mathbb{Q}_p$ test

$(a_n f_n)_n$

$1 + (p-1) = p$

$$\mathbb{Q}_p = \{0\} \cup \left\{ a_{-k} \neq 0 \;\middle|\; a_{-k} p^{-k} + a_{-k+1} p^{-k+1} + \dots \right\}$$

$a_i \in \{0, \dots, p-1\}$

$\frac{1}{q_n}$

$$\mathbb{Z}_p^\times = \{ x \in \mathbb{Q}_p \mid |x|_p = 1 \}$$

$$\left|\frac{1}{x}\right|_p = 1 \Rightarrow \frac{1}{x} \in \mathbb{Z}_p .$$

$$\mathbb{Z}_p \setminus \mathbb{Z}_p^\times = p\mathbb{Z}_p \Rightarrow p \text{ täpphtiväsei.}$$

$$\mathbb{Q}_p \setminus \{0\} = \bigcup_{z \in \mathbb{Z}} p^z \mathbb{Z}_p^\times$$