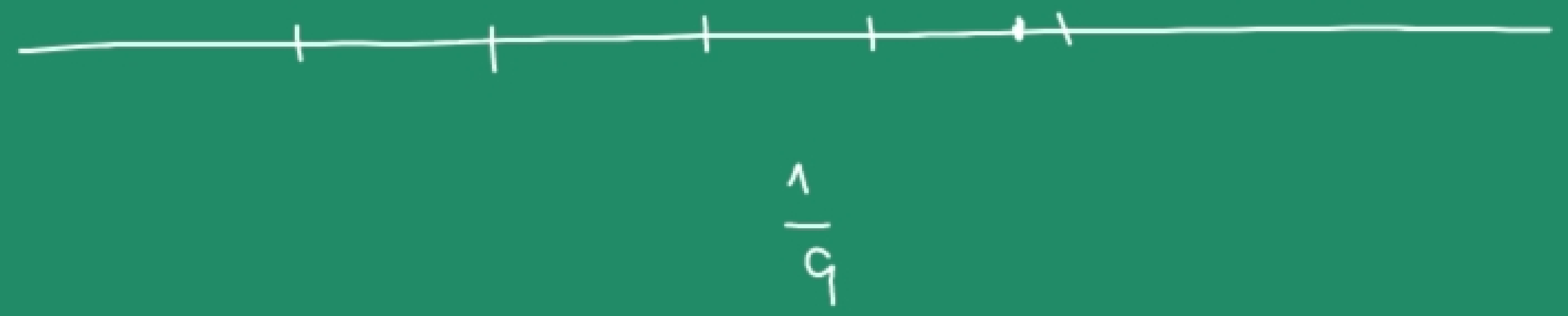
Áll: $\alpha \in \mathbb{R}$ $\quad q \in \mathbb{Z}_{>0}$ $\quad \exists\, p \in \mathbb{Z}$ $\quad \left|\alpha - \frac{p}{q}\right| \le \frac{1}{2q}$.

Biz.:

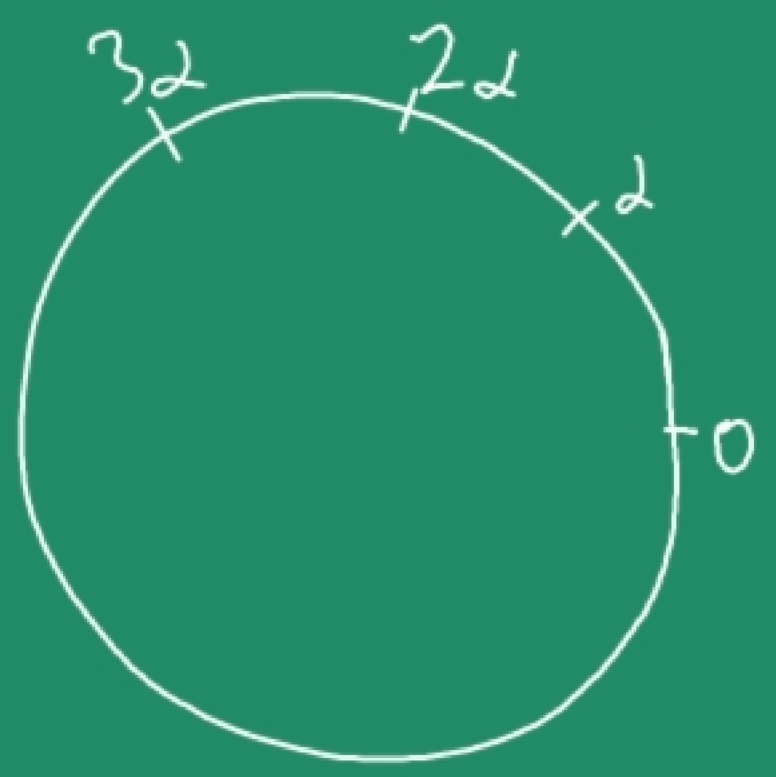


$\square$

Megj: $\alpha = \frac{p_0}{q_0} \neq \frac{p}{q} \implies \left|\alpha - \frac{p}{q}\right| \ge \frac{1}{q_0 q}$

$q_0, q > 0$

$\alpha \in \mathbb{R} \setminus \mathbb{Q}$



$0, \alpha, 2\alpha, 3\alpha, \dots, (n-1)\alpha$

$\exists \quad 0 < q < n \qquad q \in \mathbb{Z}$

$|q\alpha - p| < \frac{1}{n}$

$\left|\alpha - \frac{p}{q}\right| < \frac{1}{nq} < \frac{1}{q^2}$

<u>Dirichlet approximációs tétele</u>: $\forall \alpha \in \mathbb{R} \setminus \mathbb{Q}$ $\exists \infty$ sok különböző $\frac{p}{q} \in \mathbb{Q}$ $\quad \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$

Adott $\frac{p}{q}$ csak véges sok q-hoz jó.

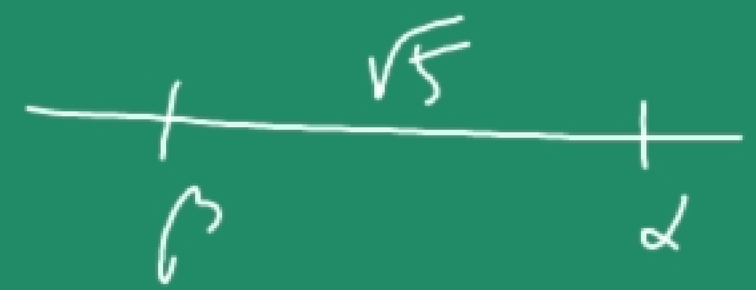
<u>Biz</u>: $\forall n \ \exists q$. Adott q csak véges sok n-hez jó.

$q\alpha \notin \mathbb{Z} \quad \delta = \mathrm{dist}(q\alpha, \mathbb{Z}) > 0 \qquad n < \frac{1}{\delta}$ $\square$

Áll: $\alpha = \frac{1+\sqrt{5}}{2}$ Csak véges sok $q > 0$ egészhez

$c > \sqrt{5}$ $\exists\, p \in \mathbb{Z}$ $\left|\alpha - \frac{p}{q}\right| < \frac{1}{cq^2}$

Biz: $f(x) = x^2 - x - 1 = (x-\alpha)(x-\beta)$ $\beta = \frac{1-\sqrt{5}}{2}$ $\alpha - \beta = \sqrt{5}$



$$\left|f\left(\frac{p}{q}\right)\right| \geq \frac{1}{q^2}$$

$$\parallel$$

$$\left|\alpha - \frac{p}{q}\right|\left|\beta - \frac{p}{q}\right|$$

Ha $\left|\alpha - \frac{p}{q}\right| < \frac{1}{cq^2}$, akkor

$$\left|\beta - \frac{p}{q}\right| > c$$

$$c - \frac{1}{cq^2} < \sqrt{5}$$

$$c - \sqrt{5} < \frac{1}{cq^2}$$

$$c(c-\sqrt{5}) < \frac{1}{q^2} \qquad \square$$

$$q < \frac{1}{\sqrt{c(c-\sqrt{5})}} < \infty$$

Liouville tétele: $\alpha \in \mathbb{R}$ n-edfokú alg. szám, $\frac{p}{q} \neq \alpha$

$\Longrightarrow \exists\, c > 0 \quad \forall\, p \in \mathbb{Z} \quad \forall\, q \in \mathbb{Z}_{>0}$
$$\left|\frac{p}{q} - \alpha\right| \geq \frac{c}{q^n}$$

Biz: $\exists\ 0 \neq f \in \mathbb{Z}[x]$ $\deg f = n$ $f(\alpha) = 0$

$$\left|f\left(\frac{p}{q}\right)\right| \geq \frac{1}{q^n} \quad \text{ha} \quad f\left(\frac{p}{q}\right) \neq 0$$

$$\frac{\left|f\left(\frac{p}{q}\right) - f(\alpha)\right|}{\left|\frac{p}{q} - \alpha\right|} \longrightarrow |f'(\alpha)| \qquad \frac{p}{q} \to \alpha$$

$$\liminf_{\frac{p}{q} \to \alpha} q^n \left|\frac{p}{q} - \alpha\right| \geq \frac{1}{|f'(\alpha)|}$$

$$0 < C < \frac{1}{|f'(\alpha)|}$$

ha $\frac{p}{q}$ elég közel van α-hoz

C' akkor $\left|\frac{p}{q} - \alpha\right| > \frac{C}{q^n}$

$$\left|\frac{p}{q} - \alpha\right| \geq C' \geq \frac{C'}{q^n}$$

$c < \min(C, C')$ $\square$

Def: $s \geq 1$. $\alpha \in \mathbb{R}$.

α s-edrendben közelíthető rac. számokkal, ha $\exists\, C > 0$:

∞ sok különböző $\frac{p}{q} \in \mathbb{Q}$ $\quad \left|\alpha - \frac{p}{q}\right| \leq \frac{C}{q^s}$

Áll $\alpha \in \mathbb{R} \setminus \mathbb{Q} \Rightarrow$ 2-rendben közelíthető

α n-edfokú alg, $s > n \Rightarrow$ α nem közelíthető s-edrendben

Def $\alpha \in \mathbb{R}$ Liouville szám, ha $\forall\, s \geq 1$: α s-edrendben közelíthető rac. számokkal

Köv $\forall$ Liouville-szám transzcendens

Pl (Liouville) $\qquad \alpha = \sum_{n=0}^{\infty} \frac{1}{2^{n!}}$ $\qquad$ Liouville-szám

$\frac{p}{q} = \sum_{n=0}^{N} \frac{1}{2^{n!}} \qquad q = 2^{N!}$ $\qquad$ 1850 +

$$\left|\alpha - \frac{p}{q}\right| = \sum_{n=N+1}^{\infty} \frac{1}{2^{n!}} < \frac{1}{2^{(N+1)!}} + \frac{1}{2^{(N+1)!+1}} + \dots = \frac{1}{2^{(N+1)!-1}}$$

$$\frac{1}{2^{(N+1)!-1}} = \frac{2}{q^{N+1}}$$

$s \geq 1$ $\quad \alpha$ s-edrendben közelíthető

$N+1 \geq s \qquad \left|\alpha - \frac{p}{q}\right| < \frac{2}{q^s}$

Tétel (Roth) α alg. , $s > 2 \Rightarrow$

$\Rightarrow$ α nem közelíthető s-edrendben rac. számokkal

Áll: $\lambda\left(\{\alpha \in \mathbb{R} : \exists\, s > 2 : \alpha \text{ } s\text{-edrendben közelíthető}\}\right) = 0$ HF

Tétel (Hurwitz). $\alpha \in \mathbb{R} \setminus \mathbb{Q} \Rightarrow \exists\, \infty$ sok kül. $\frac{p}{q} \in \mathbb{Q}$:

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}\, q^2}$$

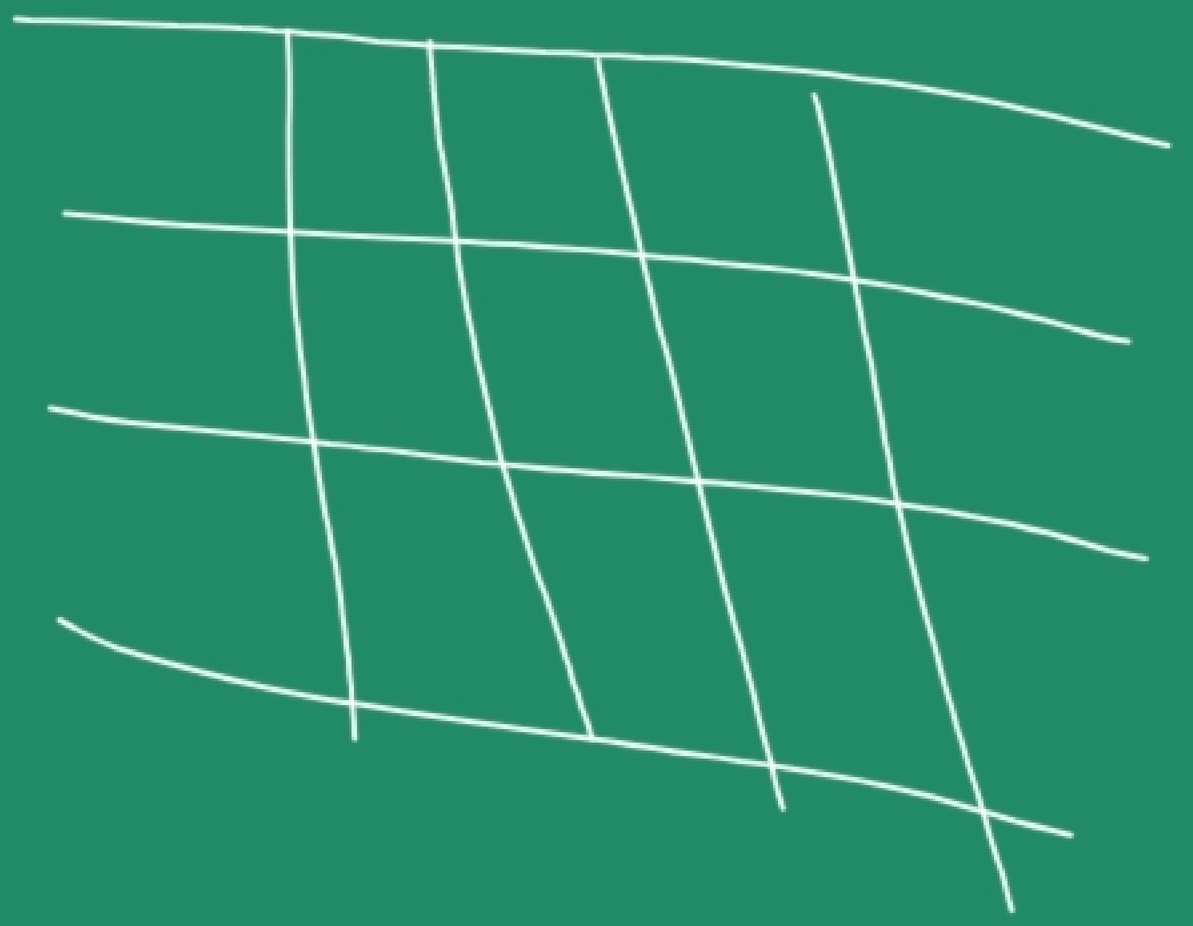
Minkowski rácsgeometriai tétele: $K \subset \mathbb{R}^d$ konvex,

$K = -K$.

Ha $|K| > 2^d$, vagy $|K| = 2^d$ és K kompakt

akkor $K \cap \mathbb{Z}^d \neq \{0\}$

Biz: $|K| > 2^d$ $\quad \exists\ x, y \in K \quad x \neq y \quad x - y \in 2\mathbb{Z}^d$



$-y \in K$

$$K \ni \frac{x + (-y)}{2} = \frac{x-y}{2} \in \mathbb{Z}^d$$

$\neq 0$

$|K| = 2^d$, K kompakt: $\forall \varepsilon > 0$

$$(1+\varepsilon)K \cap \mathbb{Z}^d \neq \{0\}$$

$\square$

Tétel: $\alpha \in \mathbb{R} \setminus \mathbb{Q} \Rightarrow \exists\, \infty$ sok $q \in \mathbb{Z}_{>0}$ $\exists\, p \in \mathbb{Z}$

$$\left|\frac{p}{q} - \alpha\right| < \frac{1}{2q^2}$$

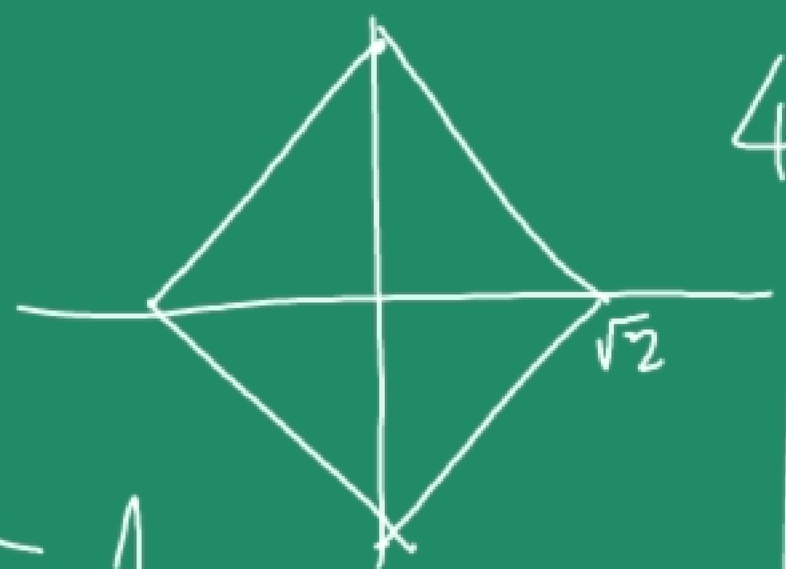
(tehát $\exists\, \infty$ sok kül. ilyen $\frac{p}{q} \in \mathbb{Q}$)

Biz: $|q|\,|p - \alpha q| \le \frac{1}{2}$ $\quad p, q \in \mathbb{Z},\ q \neq 0$

$\lambda > 0$

$$\frac{|q|}{\lambda}\,\lambda|p - \alpha q| \le \frac{1}{4}\left(\frac{|q|}{\lambda} + \lambda|p - \alpha q|\right)^2$$

$|x| + |y| < \sqrt{2}$



4 területű

$$\frac{|q|}{\lambda} + \lambda|p - \alpha q| \le \sqrt{2} \qquad p, q \in \mathbb{R}$$

$$\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1 \qquad \begin{vmatrix} 1/\lambda & 0 \\ 0 & \lambda \end{vmatrix} = 1$$

$\lambda > \sqrt{2}$ $\qquad$ $q = 0 \Rightarrow p = 0.$

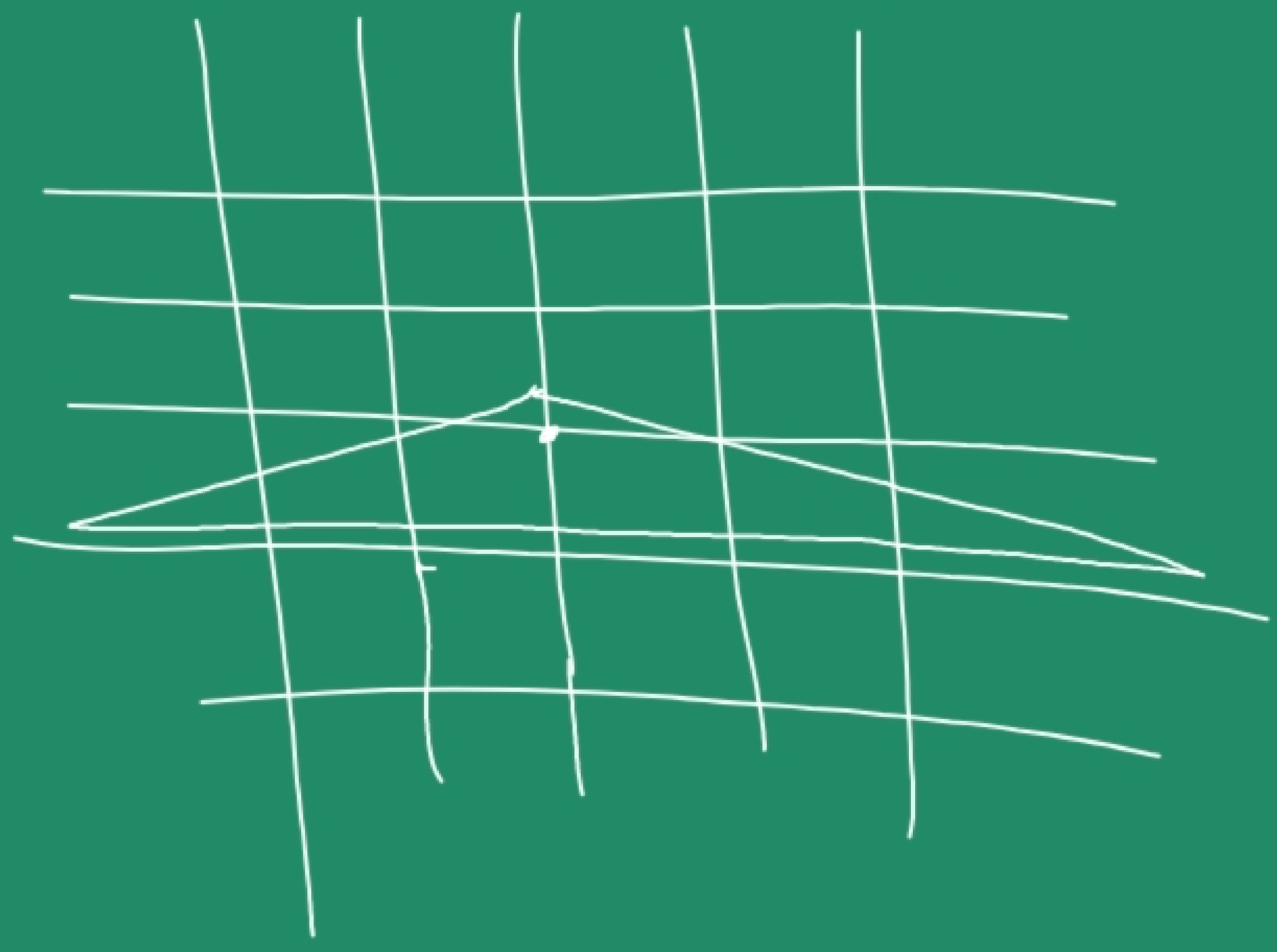
Ha p, q adott : $\lambda \leq$ korlát

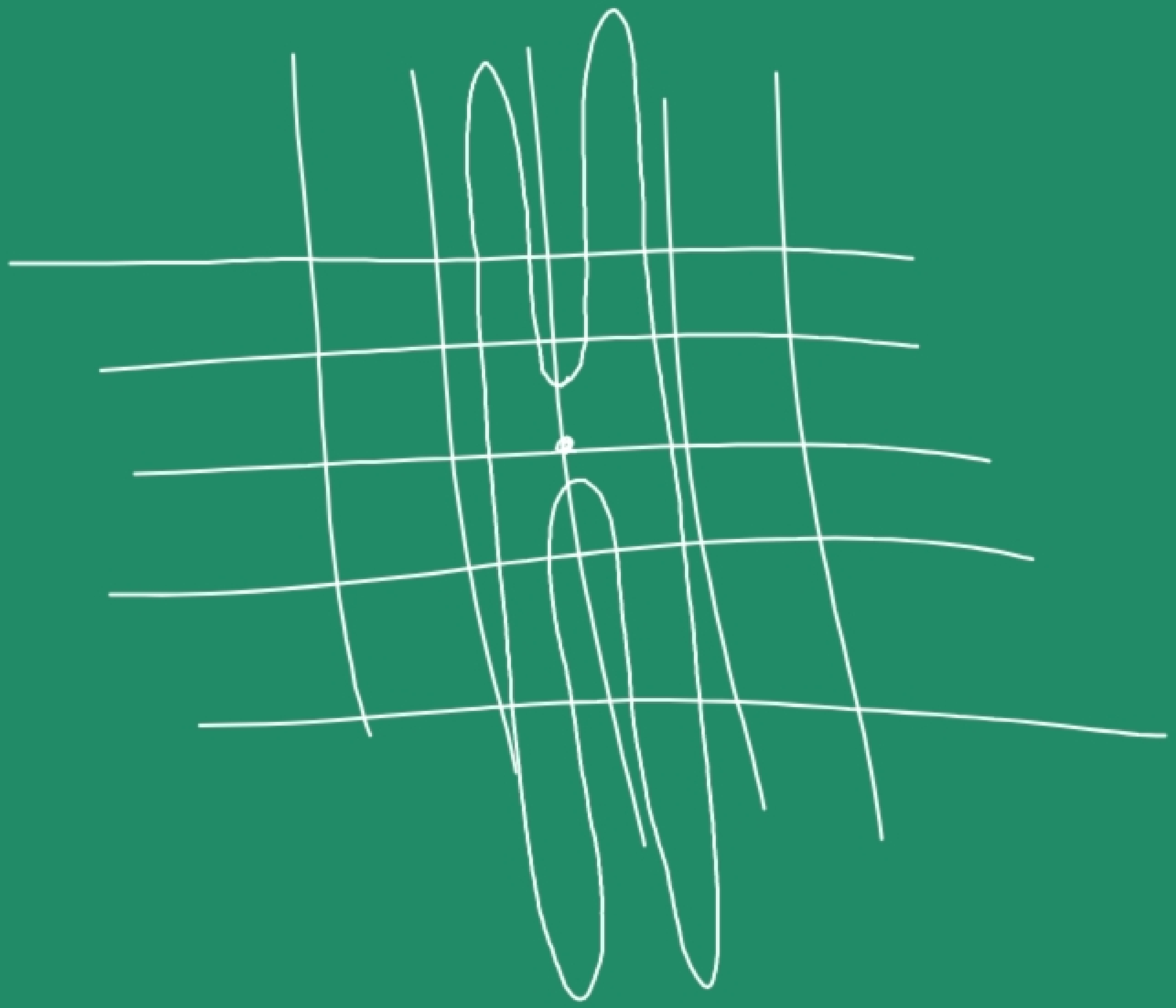
$\lambda \to \infty$: $\exists\ \infty$ sok (p, q) $\quad q \neq 0,$

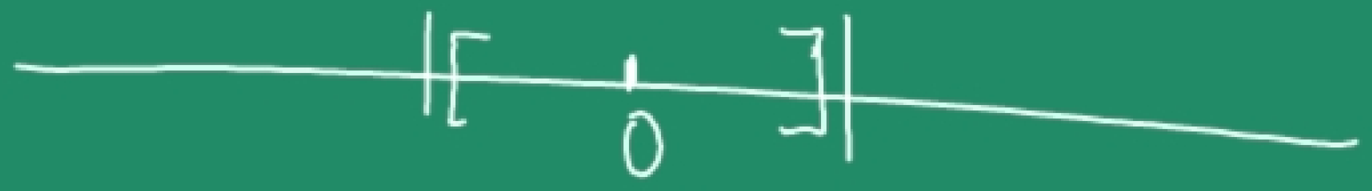
$$\left| \frac{p}{q} - \alpha \right| < \frac{1}{2q^2}$$

Adott q-hoz csak véges sok p jó. $\square$

$K = -K$ helyett $0 \in K$





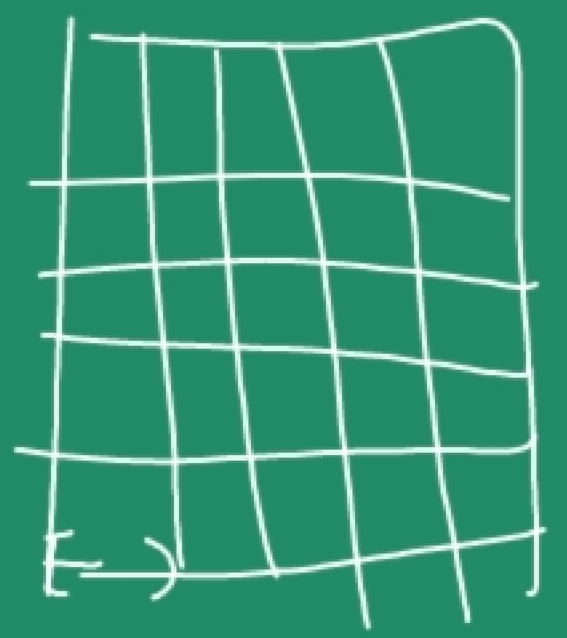


Szimultán approximáció: $\alpha_1, \dots, \alpha_d \in \mathbb{R}$

$$\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$$

$0, \alpha, 2\alpha, \dots, n^d\alpha$ $\qquad$ $n^d + 1$ pont $\in \mathbb{R}^d$

$\{0\}, \{\alpha\}, \{2\alpha\}, \dots, \{n^d\alpha\}$ $\qquad$ —"— $\in \mathbb{R}^d/\mathbb{Z}^d$



n^d cella

$\exists\ 0 < q \le n^d$ egész $\exists\ p \in \mathbb{Z}^d$

$$|q\alpha_i - p_i| < \frac{1}{n}$$

$$\left|\alpha_i - \frac{p_i}{q}\right| < \frac{1}{nq} \le \frac{1}{q^{1+\frac{1}{d}}}$$

Dirichlet szimultán approx. tétele: $\alpha_1, \dots, \alpha_d \in \mathbb{R} \setminus \mathbb{Q}$

$\Rightarrow \exists \infty$ sok $q > 0$ egész $\exists\, p_1, \dots, p_d \in \mathbb{Z}$

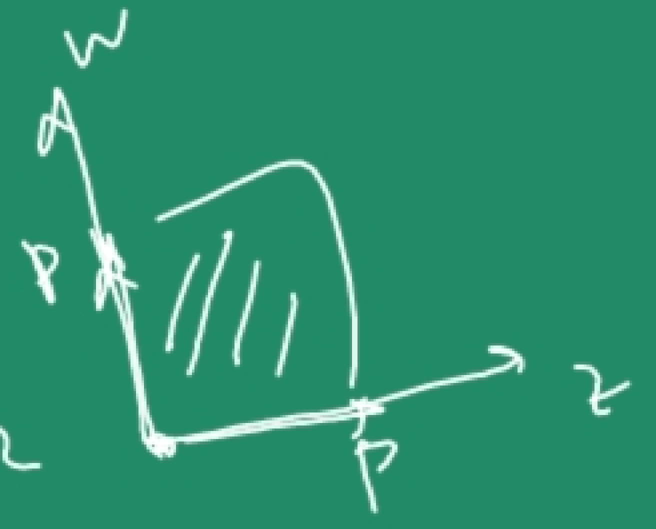
$$\forall i \qquad \left|\alpha_i - \frac{p_i}{q}\right| < \frac{1}{q^{1+\frac{1}{d}}}$$

Biz.: $\forall n \quad \exists q$

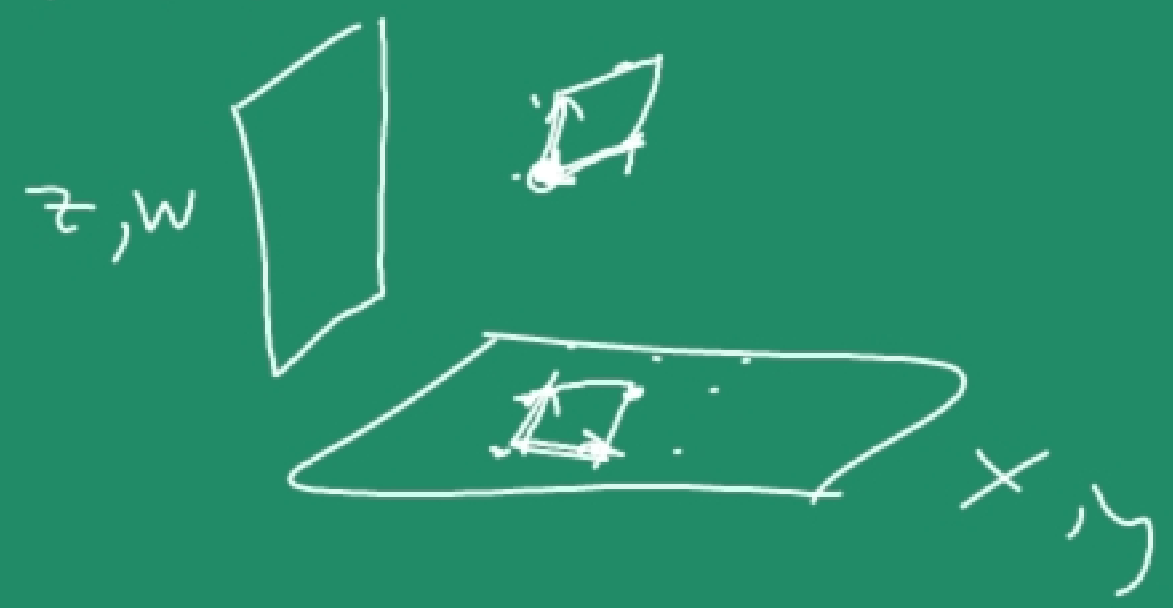
Adott $q, p_1, \dots, p_d$ -hez csak véges sok n jó

Adott q-hoz csak véges sok $p_1, \dots, p_d$ jó. $\square$

p prím $\quad p \equiv 1\ (4) \implies \exists\, x, y \in \mathbb{Z} \quad x^2+y^2=p$



<u>Tétel</u>: p prím $\implies \exists\, x, y, z, w \in \mathbb{Z} \quad p = x^2+y^2+z^2+w^2$



<u>Lemma</u>: $\exists\, a, b \in \mathbb{Z} \quad a^2+b^2+1 \equiv 0 \ (p)$

<u>Biz</u>: ✓

<u>Biz</u>:

$$z \equiv ax+by \ (p)$$
$$w \equiv bx-ay \ (p)$$

$$z^2+w^2 = a^2x^2+b^2y^2+b^2x^2+a^2y^2 = \underbrace{(a^2+b^2)}_{\equiv -1\ (p)}(x^2+y^2)$$

$$\underset{=}{p} \mid \underset{\substack{\wedge \\ 2p}}{x^2+y^2+z^2+w^2}$$

$$\left\{ (x,y,z,w) \in \mathbb{Z}^4 : \quad \right\}$$

parallelepipedon-rács, alap par. térfogata p^2

K: 0 körüli $\sqrt{2p}$ sugarú nyílt gömb

$$\frac{\pi^2}{2}\sqrt{2p}^4 = |K| > 16p^2 \qquad \pi^2 > 8$$