Kombinatorikus számelmélet

$$1 \leq a_1 < a_2 < \dots < a_k \leq n$$

Tetszőleges $S \neq T \subseteq \{1, \dots, k\}$

$$\sum_{i \in S} a_i \neq \sum_{j \in T} a_j$$

Tétel (Erdős–Moser) $k \leq \log_2 n + \log_2 \log_2 n + 1$

$n > 8 \Rightarrow$

$$k \leq \log_2 n + \frac{1}{2} \cdot \log_2 \log_2 n + 2.$$

Biz.: S $\quad 2^{k}$-féle lehet $\quad \sum_{i \in S} a_i \leq k n$

$$\Rightarrow 2^{k} \leq k \cdot n \qquad k \leq n$$

$$\boxed{k} \leq \log_2 n + \log_2 k \leq \boxed{2 \log_2 n}$$

$$\Rightarrow k \leq \log_2 n + \log_2 \log_2 n + 1$$

η : val. változó. $\quad \eta = \sum_{i=1}^{k} \xi_i \qquad \xi_i$ $\frac{1}{2}$ val.-gel a_i

$\frac{1}{2}$ val.-gel 0

független.

$$\mathbb{E}(\eta) = \frac{1}{n}\sum_{i=1}^{n} a_i$$

$$D^2(\eta) = \mathbb{E}\left(|\eta - E(\eta)|^2\right) = \mathbb{E}(\eta^2) - \mathbb{E}(\eta)^2$$

$$\left(\eta - \mathbb{E}(\eta)\right)^2 = \eta^2 - 2\eta\,\mathbb{E}(\eta) + \mathbb{E}(\eta)^2$$

X, Y ftlen $\Rightarrow$ $\mathbb{E}(XY) = \mathbb{E}(X)\,\mathbb{E}(Y)$

$$D^2(X+Y) = \mathbb{E}\left((X+Y)^2\right) - \left(\mathbb{E}(X+Y)\right)^2 = D^2(X) + D^2(Y).$$

$$D^2(\xi_i) = \mathbb{E}(\xi_i^2) - \mathbb{E}(\xi_i)^2 = \frac{a_i^2}{4}$$

$$\mathbb{E}(\xi_i^2) \rightsquigarrow \frac{a_i^2}{2}$$

$$\mathbb{E}(\xi_i) = \frac{a_i}{2}$$

$$D^2(\eta) = \sum_{i=1}^{\varepsilon} \frac{a_i^2}{4}.$$

Csebisev-egyenlőtlenség:

$$P\left(|\eta - \mathbb{E}(\eta)| \geq c\right) \leq \frac{D^2(\eta)}{c^2} = \frac{\sum_{i=1}^{\varepsilon} a_i^2}{4 \varepsilon n^2} < \frac{1}{4}$$

$$c = \sqrt{\varepsilon}\, n$$

$$\frac{3\cdot 2^{k}}{4} \le 2\sqrt{k}\, n$$

$$2^{k} \le \frac{8\sqrt{k}\, n}{3}$$

$$k \le \log_2 n + \frac{1}{2}\log_2 k + \log_2 \frac{8}{3}.$$

$$n > 8 \Rightarrow k \le \frac{3}{2}\log_2 n \Rightarrow \qquad \square$$

<u>Tétel</u> $0 < a_1 < a_2 < \dots < a_k \in \mathbb{Z}$. $\forall\, S \neq T \subseteq \{1, \dots, k\}$, $\sum_{i\in S} a_i \neq \sum_{j\in T} a_j \Rightarrow \sum_{i=1}^{k} \frac{1}{a_i} < 2.$

Biz.:

$$\prod_{i=1}^{q}\left(1+x^{a_i}\right)=\sum_{S\subseteq\{1,\dots,q\}} x^{\sum_{i\in S} a_i} < \sum_{j=0}^{\infty} x^{j}=\frac{1}{1-x}$$

$x>0$

$$\int_0^1 \frac{\log\left(\prod_{i=1}^{q}\left(1+x^{a_i}\right)\right)}{x}\,dx < \int_0^1 \frac{\log\left(\frac{1}{1-x}\right)}{x}\,dx$$

$$\| $$

$$\sum_{i=1}^{q}\int_0^1 \frac{\log\left(1+x^{a_i}\right)}{x}\,dx=\sum_{i=1}^{q}\int_0^1 \frac{\log(1+y)}{a_i\, y}\,dy=$$

$y=x^{a_i}$ $\quad dy=a_i x^{a_i-1}dx$

$dx=\frac{1}{a_i x^{a_i-1}}dy$

$$= \left(\sum_{i=1}^{r} \frac{1}{a_i} \right) \int_0^1 \frac{\log(1+y)}{y}\, dy < \int_0^1 \frac{-\log(1-x)}{x}\, dx$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \qquad \qquad \begin{array}{c} \| \\ \int_0^1 \left(1 + \frac{x}{2} + \frac{x^2}{3} + \dots \right) dx \end{array}$$

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \qquad = 1 + \frac{1}{4} + \frac{1}{9} + \dots = \zeta(2) = \frac{\pi^2}{6}$$

$$\left(\sum_{i=1}^{r} \frac{1}{a_i} \right) \underbrace{\left(1 - \frac{1}{4} + \frac{1}{9} - \qquad \right)}_{\frac{\pi^2}{12}} \Rightarrow \sum_{i=1}^{r} \frac{1}{a_i} < 2. \ \square$$

<u>Tétel</u> A nemnegatív egészek nem állnak elő véges sok csupa kül. differenciájú számtani sorozat diszjunkt uniójaként.

<u>Biz.:</u> Tf.: $\mathbb{Z}^{\geq 0} = \bigcup_{i=1}^{k} \left(a_i + d_i \mathbb{Z}^{\geq 0} \right)$ $\quad d_1 < d_2 < \dots < d_k$

$$1 + z + z^2 \dots = \sum_{i=1}^{k} \left(z^{a_i} + z^{a_i + d_i} + \dots \right)$$

$$\frac{1}{1-z} = \sum_{i=1}^{k} z^{a_i} \frac{1}{1 - z^{d_i}}$$

ε: prim. d_k-adik egységgyök

$z \to \varepsilon$ $\quad |z| < 1$.

$\lightning$ $\square$

Összeghalmaz $\mathbb{F}_p$-ben

Tétel (Cauchy–Davenport (–Chowla))

p prím $A, B \subseteq \mathbb{F}_p$ $|A| = k > 0$, $|B| = r > 0$.

$$\Longrightarrow \quad |A+B| \geq \min(p, k+r-1)$$

$$A+B = \{a+b \mid a \in A,\ b \in B\}.$$

Mj.: éles.

$$A = \{0, 1, \ldots, k-1\}$$
$$B = \{0, \ldots, r-1\}$$

Biz.:

Lemma K tetsz. test, $A, B \subseteq K$, $|A| = r$, $|B| = r$, $f(x,y) \in K[x,y]$ $\deg_x f < r$, $\deg_y f < r$ és $f(a,b) = 0$ $\forall a \in A$ $\forall b \in B$

$\Rightarrow f = 0$.

Biz.:

$$f(x,y) = h_0(x) + h_1(x)y + \dots + h_{r-1}(x)y^{r-1}$$

$a \in A$ $\quad g_a(y) = h_0(a) + h_1(a)y + \dots + h_{r-1}(a)y^{r-1}$

g_a-nak $\forall b \in B$ gyöke $\Rightarrow \equiv 0$.

$h_i(a) = 0 \quad \forall a \in A \quad \forall i = 0, \dots, r-1$

$\deg h_i < k \Rightarrow h_i = 0$ $\square$

Tph. $A, B \subseteq \mathbb{F}_p$, $|A| = k$, $|B| = r$, $A + B =: C$

$|C| \leq k + r - 2 < p$.

$$f_1(x,y) = (x+y)^{r+v-2-|C|} \cdot \prod_{c \in C} (x+y-c)$$

$$\in \mathbb{F}_p[x,y].$$

$$f_1\Big|_{A\times B} \equiv 0$$

$$\deg f_1 = r+v-2$$

$\forall i \geq r$

$u_i(x) \in \mathbb{F}_p[x]$ interpolálja az A halmazon az x^i fv.-t. $\deg u_i \leq r-1$.

$u_i(a) = a^i \quad \forall a \in A.$

x^i $\xleftrightarrow{f_1\text{-ben}}$ $u_i(x)$.

$\forall j \geq v \quad v_j(y) \in \mathbb{F}_p[y]$

$v_j(b) = b^j \quad \forall b \in B$

$f(x,y)$: így kapott polinom.

$$f(a,b) = f_1(a,b) = 0 \quad \forall a \in A,\ b \in B.$$

$$\deg_x f < k, \ \deg_y f < r.$$

Lemma $\Longrightarrow$ $f = 0$. DE! : $x^{k-1} y^{r-1}$ eh.-ja $\neq 0$. Ez uaz, mint f_1-beli eh., mert csak $< k+r-2$-fokú új tag keletkezik a cserénél.

f_1-ben:

$$f_1(x,y) = (x+y)^{k+r-2-|C|} \prod_{c\in C}(x+y-c)$$

$$(x+y)^{k+r-2} \text{-ben } x^{k-1} y^{r-1} \text{ eh.-ja}$$

$$\binom{k+r-2}{k-1} \neq 0. \qquad k+r-2<p.$$

$\square$