# Algebraic Number Theory

## Problem sheet 6

1. (3 points) Assume that $L/K$ is a finite Galois extension of number fields ($K/\mathbb{Q}$ finite) with non-cyclic Galois group. Show that there are at most finitely many primes $\mathfrak{p}$ in $K$ with only one prime divisor in $L$.

2. (3 points) Let $K/\mathbb{Q}$ be a Galois extension with non-abelian Galois group. Show that no primes $p$ are inert in $\mathcal{O}_K$.

3. (5 points) Let $K/\mathbb{Q}$ be a finite extension. Show that there are infinitely many primes $p$ that split completely in $\mathcal{O}_K$.

4. (3 points) Let $L/K$ be a finite extension of number fields and let $L \leq F$ be the Galois closure of $L$. Put $G = \mathrm{Gal}(F/K)$, $H = \mathrm{Gal}(F/L)$ and $G_P \leq G$ for the decomposition subgroup of a prime $P \lhd \mathcal{O}_F$ dividing $\mathfrak{p} \lhd \mathcal{O}_K$. Establish a natural bijection between the primes in $L$ above $\mathfrak{p}$ and the double cosets $H \backslash G / G_P$. Using this give a new proof of the fact that a prime splits completely in a finite extension if and only if it splits completely in its Galois closure. (+3points)

5. (5 points) Let $L/K$ be a – not necessarily Galois – solvable extension of number fields of degree $p$ where $p$ is a prime (ie. the Galois group of the Galois closure of the extension is solvable). Assume further that the prime $\mathfrak{p} \lhd \mathcal{O}_K$ does not ramify in $L$ and has at least two distinct prime divisors in $L$ of inertia degree 1. Show that $\mathfrak{p}$ splits completely in $L$. (Hint: you may use without proof Galois's theorem stating that whenever $G$ is a transitive solvable permutation group of prime degree then any element of $G$ different from the identity has at most 1 fixed point.)

6. (4 points) Let $A$ be a finite abelian group. Verify that there exists a finite Galois extension $L/\mathbb{Q}$ such that $\mathrm{Gal}(L/\mathbb{Q}) \cong A$. (The statement is true for any finite solvable group (a theorem of Shafarevich) but open for general finite groups.)

7. (3 points) Let $n$ be odd. Describe all quadratic extensions of $\mathbb{Q}$ contained in the cyclotomic field $\mathbb{Q}(\zeta_n)$.

8. (3 points) Let $d \in \mathbb{Z}$ be squarefree. Show that there exists a positive integer $n$ such that $\mathbb{Q}(\sqrt{d}) \subseteq \mathbb{Q}(\zeta_n)$.

9. (3 points) Show that for $q \geq 3$ the quadratic subfields in $\mathbb{Q}(\zeta_{2^q})$ are exactly $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{2})$, and $\mathbb{Q}(i\sqrt{2})$.