Algebraic Number Theory

Problem sheet 5

1. (*a*) (2 points) Verify directly that an odd prime p ramifies in $\mathbb{Q}(\sqrt{d})$ if and only if $p \mid d$ (d squarefree). Further, it is inert if $\left(\frac{d}{p}\right) = -1$ and splits into the product of two distinct primes if $\left(\frac{d}{p}\right) = 1$.

   (*b*) (2 points) What happens if $p = 2$?

2. Decompose 7 and 11 in the ring of integers of $\mathbb{Q}(\sqrt[3]{2})$ (1+1 points), respectively of $\mathbb{Q}(\alpha)$ (2+2 points) where $\alpha^3 - \alpha - 4 = 0$ (you may use the description of the ring of integers from a previous example sheet).

3. (3 points) Let f_n be the n-th element of the Fibonacci sequence ($f_0 = 0$, $f_1 = 1$, $f_{n+1} = f_{n-1} + f_n$). Show $f_p \equiv \left(\frac{p}{5}\right) \pmod{p}$ for all prime numbers $p \neq 2, 5$.

4. (2 points) Let $I, J \lhd A$ be ideals in the Dedekind domain A and let B be the integral closure of A in the finite separable field extension L/K where K is the field of fractions of A. Verify that $I = IB \cap A$ and $I \mid J \Leftrightarrow IB \mid JB$.

5. (3 points) Give an example of a pair $\mathbb{Q} \leq K_1, K_2$ of finite extensions and prime p such that p ramifies complete in both extensions K_1 and K_2 (ie. $r = 1 = f_1$) but not in the composite extension K_1K_2. (Hint: You may even choose quadratic extensions.)

6. (*a*) (3 points) Assume $\mathcal{O}_K = \mathbb{Z}[\alpha]$ for some $\alpha \in \mathcal{O}_K$ and let $p \in \mathbb{Z}$ be a prime. Show that there are at most p prime ideals $\mathfrak{p}_i \lhd \mathcal{O}_K$ dividing p with $f_i = 1$ (ie. $\mathcal{O}_K/\mathfrak{p}_i \cong \mathbb{F}_p$).

   (*b*) (2 points) Show that 3 splits completely in $K = \mathbb{Q}(\sqrt{7}, \sqrt{10})$, ie. by part (*a*) there does not exist $\alpha \in K$ with $\mathcal{O}_K = \mathbb{Z}[\alpha]$.

   (*c*) (3 points) Find an integer $0 < N \in \mathbb{Z}$ and an element $\alpha \in \mathcal{O}_K$ such that $\mathcal{O}_K[1/N] = \mathbb{Z}[1/N][\alpha]$ and decompose all primes $p \mid N$ as a product of prime ideals in $\mathcal{O}_K$.

7. (4 points) Let $\mathbb{Q} \leq K$ be a finite extension. Show that a prime $\mathfrak{p} \lhd \mathcal{O}_K$ splits completely in $(K \leq)L$ and in $(K \leq)L'$ then it splits completely in the composite extension LL', as well (ie. in the smallest subfield of a given algebraic closure $\overline{K}$ containing both L and L').

8. (3 points) Let $\mathbb{Q} \leq K \leq L$ be finite extensions and $K \leq L \leq F$ be the Galois closure of L/K. Show that a prime $\mathfrak{p} \lhd \mathcal{O}_K$ splits completely in L if and only if it splits completely in F.

9. (3 points) Show that no primes of $\mathbb{Q}(\sqrt{-5})$ ramify in the extension $\mathbb{Q}(\sqrt{-1}, \sqrt{-5})/\mathbb{Q}(\sqrt{-5})$.