Algebraic Number Theory

Problem sheet 3

1. (3 points) Verify the Chinese Remainder Theorem for Dedekind domains: If $A \lhd \mathcal{O}$ is an ideal in the Dedekind domain $\mathcal{O}$ with decomposition $A = P_1^{\nu_1} \dots P_t^{\nu_t}$ as a product of prime ideals then we have
$$\mathcal{O}/A \cong \bigoplus_{i=1}^{t} \mathcal{O}/P_i^{\nu_i} .$$

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2. (3 points) Show that a lattice $\Gamma \subset V$ is complete if and only if V/Γ is compact.

3. (2 points) Let $L_i(x_1, \dots, x_n) = \sum_{j=1}^{n} a_{ij}x_j$ $(i = 1, \dots, n)$ be real homogeneous linear polynomials such that $\det((a_{ij}))_{ij} \neq 0$ and let $c_1, \dots, c_n$ be positive real numbers with $c_1 \dots c_n > |\det((a_{ij}))_{ij}|$. Verify that there exist not all zero integers $m_1, \dots, m_n \in \mathbb{Z}$ such that $|L_i(m_1, \dots, m_n)| < c_i$ $(i = 1, \dots, n)$.

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4. (2+2 points) Show that the map
$$\begin{array}{rcl} \mathbb{C} \otimes_{\mathbb{Q}} K & \to & K_{\mathbb{C}} \\ z \otimes \alpha & \mapsto & z \cdot (j\alpha) \end{array}$$
is an isomorphism of rings. Moreover, prove that its restriction to $\mathbb{R} \otimes_{\mathbb{Q}} K$ also induces an isomorphism between $K \otimes_{\mathbb{Q}} \mathbb{R}$ and $K_{\mathbb{R}}$.

5. (1 point each) Verify that the ring of integers of quadratic extensions of $\mathbb{Q}$ with discriminant $5, 8, 11, -3, -4, -7, -8, -11$, respectively, are all unique factorization domains.

6. (*a*) (4 points) Let $X_t := \{z \in K_{\mathbb{R}} \mid \sum_\tau |z_\tau| < t\}$ for any real number $0 < t$. Verify $\mathrm{vol}(X_t) = 2^r \pi^s \frac{t^n}{n!}$.

   (*b*) (4 points) Assume $|K/\mathbb{Q}| = n$. Show $|d_K|^{1/2} \geq \frac{n^n}{n!} \left(\frac{\pi}{4}\right)^{n/2}$. (Hint: choose t so that $\mathrm{vol}(X_t) > 2^n \mathrm{vol}(\Gamma)$ where $\Gamma = j(\mathcal{O}_K) \subset K_{\mathbb{R}}$. Apply the inequality between geometric and arithmetic mean on the numbers $|\tau(\alpha)|$ where $0 \neq \alpha \in \mathcal{O}_K$ is the element with $j(\alpha) \in X_t$ guaranteed by Minkowski's lattice point theorem. Finally, note that $1 \leq |N_{K/\mathbb{Q}}(\alpha)|$.)

   (*c*) (1 point) Show that whenever the degree $|K : \mathbb{Q}|$ goes to infinity, so does the discriminant d_K. Moreover, we have $d_K > 1$ for all extensions $K \neq \mathbb{Q}$.

7. (*a*) (3 points) Let $A \lhd \mathcal{O}_K$ be an ideal whose class has order m in the class group (ie. m is the smallest integer such that $A^m = (\alpha)$ is a principal ideal). Prove that $A\mathcal{O}_L = (\beta)$ where $L = K(\beta)$ and $\beta^m = \alpha$.

(*b*) (1 point) Verify that for each number field K there is a finite extension L/K in which all ideals of $\mathcal{O}_K$ become principal (ie. all ideals of $\mathcal{O}_K$ *capitulate* in $\mathcal{O}_L$).

(*c*) (2 points) Verify that the ring Ω of all algebraic integers (in $\mathbb{C}$) is a *Bézout domain*, ie. all finitely generated ideals are principal (but not noetherian, so not a PID).

8. (*a*) (3 points) Show that over a Bézout domain every finitely generated torsion-free module is free.

   (*b*) (2 points) Verify that $\bigcup_{n=1}^{\infty} \mathbb{C}[[x^{1/n}]]$ is a Bézout domain.

9. In this exercise we compute the rank of the unit group $\mathcal{O}_K^\times$ as an abelian group using a multiplicative version of Minkowski's theory.

   (*a*) (3 points) Let $K_\mathbb{C}^\times = \prod_\tau \mathbb{C}^\times$ be the multiplicative group of invertible elements in the ring $K_\mathbb{C}$ and let $N\colon K_\mathbb{C}^\times \to \mathbb{C}^\times$ be the group homomorphism defined as the product of coordinates. Further define the homomorphism $l := \log|\cdot|\colon K_\mathbb{C}^\times \to \prod_\tau \mathbb{R}$ coordinatewise. Show that the kernel of $l \circ j\colon \mathcal{O}_K^\times \to \prod_\tau \mathbb{R}$ is the torsion subgroup in $\mathcal{O}_K^\times$, ie. the group $\mu(K)$ of roots of unity in K.

   (*b*) (2 points) Verify that for each $\alpha \in \mathcal{O}_K^\times$ the element $l \circ j(\alpha) \in \prod_\tau \mathbb{R}$ is fixed by the complex conjugation (permuting $\tau \in \mathrm{Hom}_\mathbb{Q}(K, \mathbb{C})$). Further show that the sum of coordinates of $l \circ j(\alpha)$ equals 0.

   (*c*) (5 points) Show that $l \circ j(\mathcal{O}_K^\times)$ is a full lattice in the subspace H where

$$H = \{x_\tau \in \prod_\tau \mathbb{R} \mid \sum_\tau x_\tau = 0 \text{ and } x_{\sigma_k} = x_{\overline{\sigma_k}}\ (k = 1, \ldots, s)\} .$$

   In particular, $\mathcal{O}_K^\times \cong \mu(K) \times \mathbb{Z}^{r+s-1}$ (as an abelian group).