Algebraic Number Theory

Problem sheet 2

1. (3 points) Find an integral basis in the ring of algebraic integers in $\mathbb{Q}(\sqrt{d})$ where $d \in \mathbb{Z}$ is squarefree.

2. (3 points) Let $\mathcal{O}$ be the ring of integers in $\mathbb{Q}(\sqrt{d})$ ($d \in \mathbb{Z}$ squarefree) and $p \nmid 2d$ be a prime. Show that $p\mathcal{O}$ is a prime ideal if and only if $\left(\frac{d}{p}\right) = -1$.

3. (4 points) Verify that the ring $\mathbb{C}[x,y]/(y^2 - x^3)$ is not integrally closed.

4. (5 points) Let $K/\mathbb{Q}$ be finite (ie. K is a *number field*) and d_K be the discriminant of K. Show that $d_K \equiv 0, 1 \pmod 4$. (Hint: Try taking the permanent (ie. sum of all expression with $+$ sign as in the determinant) of the matrix $((\sigma_i \alpha_j))$ and use the identity $(a-b)^2 = (a+b)^2 - 4ab$.)

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5. (2 points) Decompose $33 + 11\sqrt{-7}$ as a product of irreducible elements in the ring of integers in $\mathbb{Q}(\sqrt{-7})$.

6. (4 points) Show that a Dedekind domain with finitely many prime ideals is a principal ideal domain.

7. (3+3 points)

   (i) Let $\mathcal{O}$ be a Dedekind domain and $I \lhd \mathcal{O}$ be an ideal. Show that any ideal in $\mathcal{O}/I$ is principal.

   (ii) Show that any ideal in a Dedekind domain is generated by two elements.

8. (5 points) Let $\mathcal{O}$ be an integral domain in which there is a unique factorization of ideals as a product of prime ideals. Show that $\mathcal{O}$ is a Dedekind domain (ie. noetherian, integrally closed, and 1-dimensional).

9. (5 points) Show that each ideal in a Dedekind domain is a projective module.

10. (3+3 points) Let $f(x,y) \in \mathbb{C}[x,y]$ be an irreducible nonsingular polynomial (ie. $(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (1)$ as ideals in $\mathbb{C}[x,y]$). Verify that $\mathbb{C}[x,y]/(f(x,y))$ is a Dedekind domain. What is the class group of $\mathbb{C}[x,y]/(f(x,y))$ if we further assume that f is of the form $f(x,y) = y^2 - x^3 - ax - b$ where $x^3 + ax + b$ has no repeated roots (ie. f is the equation of an elliptic curve)? (Compare to Problem 3 where the curve is *singular*.)