Algebraic Number Theory

Problem sheet 10

1. (2 points) Let $K/\mathbb{Q}_p$ be a finite extension. Verify that the power series $\log(1+x) = x - \frac{x^2}{2} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \ldots$ is convergent on the maximal ideal of the valuation ring of K.

2. (3 points) For what p-adic absolute value of x does the power series $\exp(x) = 1 + x + \cdots + \frac{x^n}{n!} + \ldots$ converge? (Determine the radius of convergence.)

3. (3 points) Let $K/\mathbb{Q}_p$ be a finite extension with absolute ramification index e and maximal ideal $\mathfrak{p}$. Show that $\exp\colon \mathfrak{p}^n \to U^{(n)} = 1 + \mathfrak{p}^n$ and $\log\colon U^{(n)} \to \mathfrak{p}^n$ establish an isomorphism between the additive group $\mathfrak{p}^n$ and the multiplicative group $U^{(n)}$ assuming $n > \frac{e}{p-1}$. In particular, $U^{(n)}$ is torsion-free as an abelian group.

4. (3 points) Let $|K : \mathbb{Q}_p| = d$ and $\pi \in K$ be a prime element. Show the isomorphism $K^\times \cong \pi^{\mathbb{Z}} \oplus Z_{q-1} \oplus Z_{p^a} \oplus \mathbb{Z}_p^d$ for a suitable integer $a \geq 0$ where q denotes the cardinality of the residue field.

5. (2 points) Verify that the binomial series $(1+x)^z = \sum_{n=0}^{\infty} \binom{z}{n} x^n$ is convergent for any fixed $z \in \mathbb{Z}_p$ and $v_p(x) > \frac{1}{p-1}$. Moreover, we have $(1+x)^z = \exp(z\log(1+x))$ in this case.

6. (3 points) Let $K/\mathbb{Q}_p$ be finite. Show that any finite index subgroup of $K^\times$ is open (hence closed, too).

7. (3 points) Let $K/\mathbb{Q}_p$ be a finite extension, $\pi \in K$ be a prime element, and v_π be the normalized valuation (ie. $v_\pi(\pi) = 1$). Since K is a locally compact abelian group, it admits a translation invariant Haar measure dx, that is unique if we assume $\int_{\mathcal{O}_K} dx = 1$. Show that $|a|_\pi = \int_{a\mathcal{O}_K} dx$ where the absolute value $|\cdot|$ is normalized so that $|a|_\pi := q^{-v_\pi(a)}$ where $q = |\mathcal{O}_K/(\pi)|$ is the cardinality of the residue field. Further show that $\frac{dx}{|x|_\pi}$ is a translation invariant Haar measure on the multiplicative group $K^\times$.

8. (3 points) Let $K/\mathbb{Q}_p$ be finite. Show that the slopes of the Newton polygon of a polynomial $f(x) \in K[x]$ are exactly the π-adic valuations of its roots (with multiplicity) where $\pi \in K$ is a prime element (and the Newton polygon is constructed using the π-adic valuation).

9. (3 points) Let $K/\mathbb{Q}_p$ be finite. Assume that $f(x) \in K[x]$ is irreducible. Show that the Newton polygon of f consists of a single segment. Give examples that this segment may or may not contain a lattice point apart from the end points.