# Algebraic Number Theory

Problem Sheet 9

to be handed in until 29th November 2018

1. Let $M_i$, $i \in I$ (resp. $N_j$, $j \in J$) a direct (resp. inverse) system of abelian groups and $M, N$ be abelian groups. Under what conditions can Hom and $\otimes$ be interchanged with $\varinjlim$ and $\varprojlim$? Eg. does $\mathrm{Hom}(\varprojlim_i N_i, M) \cong \varprojlim_i \mathrm{Hom}(N_i, M)$ or $\varinjlim_i (M_i \otimes N) \cong (\varinjlim_i M_i) \otimes N$ always hold? Positive statements with proof for 3 points each, counterexamples for 2 points each. (If <u>needed</u> you may assume that the objects in question are finitely generated or even have finite cardinality.)

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In the following exercises we construct the Witt rings of perfect rings of characteristic $p$ and show that this formation is an equivalence of categories. Let $A$ be a commutative ring with 1 such that $p = \underbrace{1 + \cdots + 1}_{p} \in A$ is not a zero divisor (and nonzero) and the natural map $A \to \varprojlim_n A/p^n A$ is an isomorphism (ie. $A$ is *p-adically complete*). Assume further $R := A/pA$ is a *perfect ring of characteristic* $p$, ie. raising to the $p$th power („Frobenius endomorphism") is bijective: for each $x \in R$ there exists a unique $x^{p^{-1}} := y \in R$ with $y^p = x$. Rings $A$ with these properties are called strict $p$-rings. For example $A = \mathbb{Z}_p$ is a strict $p$-ring.

2. (2 points) Verify that the Frobenius endomorphism is always injective on a field of characteristic $p$ and it is onto if and only if $k$ is perfect (ie. no irreducible polynomial over $k$ admits multiple roots in any field extension). Give an example of an imperfect field of characteristic $p$.

3. (2 points) Show that the $p$-Frobenius on a ring $R$ of characteristic $p$ (ie. $1 \in R$ commutative and $\underbrace{1 + \cdots + 1}_{p} = 0$) is injective if and only if $R$ is reduced, ie. it has no nilpotent elements.

4. (3 points) Let $A$ be a strict $p$-ring, in particular, $R = A/pA$ is perfect of char $p$. For an element $x \in R$ we denote by $\hat{x}$ a (once and for all fixed) lift of $x$ to $A$. Show that the limit $[x] := \lim_{n\to\infty} (\widehat{x^{p^{-n}}})^{p^n}$ exists in the $p$-adic topology on $A$. Further verify $[xy] = [x][y]$. The element $[x]$ is called the multiplicative (or Teichmüller) representative of $x$.

Our next goal is to construct a strict $p$-ring $W(R)$ for any given perfect ring $R$ of characteristic $p$ such that $R \cong W(R)/pW(R)$. $W(R)$ is called the Witt ring of $R$ (or the ring of Witt vectors of $R$). The elements of $W(R)$ are formal sums of the form $\sum_{i=0}^{\infty} p^i [x_i]$ where $x_i \in R$. The expressions $[x_i]$ are going to be the multiplicative representatives of $x_i$. In order to define addition and multiplication on the set $W(R)$ we first need to construct the Witt rings of the free objects in the category of perfect rings of characteristic $p$ and investigate these operations therein. Let $X_0, X_1, \dots, Y_0, Y_1, \dots$ be two infinite sequences of formal variables and $p$ be a fixed prime. For each $0 \le n$ and $0 \le i$ consider a formal $p^n$th root $X_i^{p^{-n}}$, resp. $Y_i^{p^{-n}}$ of the variables $X_i$ and $Y_i$ (ie. these are a priori formal variables, as well, but we quotient out by the identities $(X_i^{p^{-n}})^p = X_i^{p^{-n+1}}$, resp. $(Y_i^{p^{-n}})^p = Y_i^{p^{-n+1}}$ in the polynomial ring). Put

$$\begin{aligned} \mathbb{Z}_p[X_i^{p^{-\infty}}, Y_i^{p^{-\infty}} \mid i \ge 0] &:= \bigcup_n \mathbb{Z}_p[X_i^{p^{-n}}, Y_i^{p^{-n}} \mid i \ge 0] \; ; \\ S &:= \varprojlim_n \mathbb{Z}_p[X_i^{p^{-\infty}}, Y_i^{p^{-\infty}} \mid i \ge 0]/(p^n) \; . \end{aligned}$$

5. (3 points) Verify that $S$ is a strict $p$-ring. In particular, there exist polynomials $S_i, P_i \in S/pS = \mathbb{F}_p[X_i^{p^{-\infty}}, Y_i^{p^{-\infty}} \mid i \ge 0]$ such that

$$\begin{aligned} \left(\sum_{i=0}^{\infty} p^i X_i\right) + \left(\sum_{i=0}^{\infty} p^i Y_i\right) &= \sum_{i=0}^{\infty} p^i [S_i] \\ \left(\sum_{i=0}^{\infty} p^i X_i\right)\left(\sum_{i=0}^{\infty} p^i Y_i\right) &= \sum_{i=0}^{\infty} p^i [P_i] \; . \end{aligned}$$

6. (2 points) Determine the polynomials $S_0, S_1, P_0, P_1 \in \mathbb{F}_p[X_i^{p^{-\infty}}, Y_i^{p^{-\infty}} \mid i \ge 0]$.

7. (3 points) Let $R$ be a perfect ring of char $p$ and $W(R) = \{r = (r_0, r_1, \dots) \mid r_i \in R, i \ge 0\} = R^{\mathbb{N}}$ as a set. Consider the following operations: $(r+s)_n := S_n(r_0, r_1, \dots, s_0, s_1, \dots)$ and $(rs)_n := P_n(r_0, r_1, \dots, s_0, s_1, \dots)$. Show that these operations make $W(R)$ into a strict $p$-ring.

8. (3 points) Show the following universal property of $W(R)$: If $A$ is a strict $p$-ring and $\varphi\colon R \to A/pA$ is a ring homomorphism then there exists a unique lift $\tilde{\varphi}\colon W(R) \to A$ of $\varphi$. In particular $W$ is a functor. NB: the $p$-Frobenius $\mathrm{Frob}_p\colon R \to R$ can therefore be lifted to $W(R)$ (and such a lift is called a Frobenius lift which is unique in our case $R$ being perfect).

9. (3 points) Verify that the functors $R \mapsto W(R)$ and $A \mapsto A/pA$ are quasi-inverse equivalences of categories, ie. there exist a natural isomorphisms $\Phi_R\colon R \to W(R)/pW(R)$ and $\Psi_A\colon A \to W(A/pA)$ for each perfect ring $R$ of char $p$ and strict $p$-ring $A$. Here naturality (in case of $R$ for example) means that whenever $f\colon R_1 \to R_2$ is a ring homomorphism of perfect rings of char $p$ then the diagram

$$\begin{tikzcd} R_1 \arrow[r, "f"] \arrow[d, "\Phi_{R_1}"'] & R_2 \arrow[d, "\Phi_{R_2}"] \\ W(R_1)/pW(R_1) \arrow[r, "f_*"'] & W(R_2)/W(R_2) \end{tikzcd}$$

is commutative.