Algebraic Number Theory

Problem Sheet 8

to be handed in until 22nd November 2018

1. (3 points) Let K be a field that is complete with respect to some archimedean absolute value $|\cdot|$. Show that K is (topologically) isomorphic to either $\mathbb{R}$ or $\mathbb{C}$.

2. (3 points) Let $K/\mathbb{Q}$ be a finite extension. Verify that any nontrivial absolute value on K is equivalent to either the $\mathfrak{p}$-adic valuation coming from a prime ideal in $\mathfrak{p} \lhd \mathcal{O}_K$ (non-archimedean case) or to the restriction of the usual absolute value on $\mathbb{C}$ under an embedding $\tau\colon K \to \mathbb{C}$ (archimedean case). Two such absolute values are equivalent if and only if they are both archimedean and the corresponding embeddings $\tau_1, \tau_2\colon K \to \mathbb{C}$ ($\mathbb{Q}$-homomorphisms) are complex conjugates of each other.

3. (1 point) Write -1 in p-adic form $-1 = \sum_{i=0}^{\infty} a_i p^i$.

4. (2 points) Write $2/3$ and $-2/3$ in 5-adic form.

5. (4 points) Prove that a p-adic number of the form $\sum_{i=-m}^{\infty} a_i p^i$ ($a_i = 0, 1, \dots, p-1$, $i \geq -m$) is rational if and only if its digits are eventually periodic.

6. (3 points) Solve the equation $x^2 = 2$ in $\mathbb{Z}_7$.

7. (3 points) Verify that any (algebraic) field automorphism of $\mathbb{Q}_p$ is continuous (and therefore identical).

8. (2 points) Let $n \geq 1$ be an integer and $n = a_0 + a_1 p + \dots + a_r p^r$ be its form in base p ($0 \leq a_i < p$). Further put $s = a_0 + a_1 + \dots + a_r$. Verify that $v_p(n!) = \frac{n-s}{p-1}$.

9. (2 points) Verify that the sequence $1, 1/10, 1/100, \dots, 1/10^n, \dots$ is not convergent in $\mathbb{Q}_p$ for any prime p.

10. (3 points) Let $\varepsilon \in 1 + p\mathbb{Z}_p$ and $\alpha = a_0 + a_1 p + a_2 p^2 + \dots$ be a p-adic integer, and put $s_n = a_0 + a_1 p + \dots + a_n p^n \in \mathbb{Z}$. Show that the limit $\varepsilon^\alpha := \lim_{n\to\infty} \varepsilon^{s_n}$ exists in $\mathbb{Z}_p$ making $1 + p\mathbb{Z}_p$ a $\mathbb{Z}_p$-module (written multiplicatively).

11. (3 points) Assume $(a, p) = 1$ ($a \in \mathbb{Z}$). Show that the sequence a^{p^n} converges in $\mathbb{Q}_p$.

12. (2 points) Verify that $\mathbb{Q}_p$ is not isomorphic (algebraically) to $\mathbb{Q}_q$ for primes $p \neq q$.

13. (2 points) Show that the algebriac closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ is an infinite extension of $\mathbb{Q}_p$.

14. (5 points) Denote by $\mathbb{Z}_p[[X]]$ the ring of formal power series over $\mathbb{Z}_p$. Let $g \in \mathbb{Z}_p[[X]]$ be arbitrary and $f(X) = a_0 + a_1 X + \cdots + a_n X^n + \cdots \in \mathbb{Z}_p[[X]]$ such that $p \mid a_i$ for all $0 \leq i \leq n-1$ but $p \nmid a_n$. Verify that one can divide g by f with residues in this situation, ie. there exist a power series $q \in \mathbb{Z}_p[[X]]$ and a polynomial $r \in \mathbb{Z}_p[X]$ of degree at most $n-1$ with $g = qf + r$ (and these are unique).

15. (3 points) ("p-adic Weierstraß preparation theorem") Show that you may write any $0 \neq f(X) \in \mathbb{Z}_p[[X]]$ of the form $f(X) = p^\mu g(X) u(X)$ where $\mu \geq 0$ is an integer, $u(X) \in \mathbb{Z}_p[[X]]^\times$ is a unit, and $g(X) \in \mathbb{Z}_p[X]$ is a monic polynomial with all non-leading coefficients divisible by p.