Algebraic Number Theory

Problem sheet 6

to be handed in until 8th November 2018

1. (3 points) Let L/K be a finite Galois extension of number fields (ie. $K/\mathbb{Q}$ finite) such that $\mathrm{Gal}(L/K)$ is *not* cyclic. Show that there are at most finitely many primes $\mathfrak{p} \lhd \mathcal{O}_K$ having only one prime divisor in L.

2. (3 points) Let $K/\mathbb{Q}$ be a finite Galois extension with nonabelian Galois group. Verify that no rational prime p remains prime in $\mathcal{O}_K$.

3. (3 points) Let L/K be a finite extension of number fields and let $L \leq F$ be the Galois closure. Set $G := \mathrm{Gal}(F/K)$, $H = \mathrm{Gal}(F/L)$, $G_P \leq G$ the decomposition subgroup of a prime $P \lhd \mathcal{O}_F$ above $\mathfrak{p} \lhd \mathcal{O}_K$. Construct a natural bijection between the primes in L dividing $\mathfrak{p}$ and the double cosets $H \backslash G / G_P$. Use this to show that $\mathfrak{p}$ splits completely in L if and only if it does in F. (+3points)

4. (5 points) Let $K/\mathbb{Q}$ be an arbitrary finite extension. Show that there are infinitely many primes p that split completely in $\mathcal{O}_K$.

5. (5 points) Let L/K be a – not necessarily Galois – solvable extension of number fields of prime degree p (ie. the Galois closure of L has solvable Galois group over K). Assume further that a prime $\mathfrak{p} \lhd \mathcal{O}_K$ does not ramify in L and has at least two prime divisors in L of inertia degree 1. Show that $\mathfrak{p}$ splits completely in L. (Hint: You may use the theorem of Galois stating that if G is a solvable transitive permutation group of degree p then any nontrivial element of G has at most one fixed point.)

6. (4 points) Let A be a finite abelian group. Show that there exists a finite Galois extension $L/\mathbb{Q}$ with $\mathrm{Gal}(L/\mathbb{Q}) \cong A$. (The statement is true for any solvable group by Shafarevich's theorem but is open in general (inverse Galois problem).)

7. (3 points) Let n be odd. Describe all the quadratic extensions of $\mathbb{Q}$ contained in $\mathbb{Q}(\zeta_n)$.

8. (3 points) Let $d \in \mathbb{Z}$ be square-free. Show that there exists a positive integer n such that $\mathbb{Q}(\sqrt{d}) \subseteq \mathbb{Q}(\zeta_n)$.

9. (3 points) Let $q \geq 3$. Verify that the quadratic fields contained in $\mathbb{Q}(\zeta_{2^q})$ are the following: $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(i\sqrt{2})$.