Algebraic Number Theory

5th Problem sheet

to be handed in until 25th October 2018

1. (*a*) (2 points) Show that an odd prime ramifies in $\mathbb{Q}(\sqrt{d})$ if and only if $p \mid d$ (here d is square-free). Moreover, p is inert (resp. splits) if $\left(\frac{d}{p}\right) = -1$ (resp. $\left(\frac{d}{p}\right) = 1$).

   (*b*) (2 points) What if $p = 2$?

2. Decompose 7 and 11 in the ring of integers of $\mathbb{Q}(\sqrt[3]{2})$ (1 point each) and of $\mathbb{Q}(\alpha)$ (2+2 points) where $\alpha^3 - \alpha - 4 = 0$ (You may use the desrciption of the rings of integers).

3. (3 points) Let f_n be the nth element of the Fibonacci sequence ($f_0 = 0$, $f_1 = 1$, $f_{n+1} = f_n + f_{n-1}$). Verify that $f_p \equiv \left(\frac{p}{5}\right) \pmod{p}$ if $p \neq 2, 5$ is a prime.

4. (2 points) Let $I, J \lhd A$ ideals in a Dedekind domain A and let B be the integral closure of A in a finite separable extension L/K where K is the residue field of A. Show that $I = IB \cap A$ and $I \mid J \Leftrightarrow IB \mid JB$.

5. (3 points) Give an example of finite extensions $\mathbb{Q} \leq K_1, K_2$ and a prime p such that p ramifies completely in both K_1 and K_2 (ie. $r = 1 = f_1$), but not in the composite K_1K_2. (Hint: You may choose quadratic extensions.)

6. (*a*) (3 points) Assume $\mathcal{O}_K = \mathbb{Z}[\alpha]$ for some $\alpha \in \mathcal{O}_K$ where $K/\mathbb{Q}$ is a finite extension and let $p \in \mathbb{Z}$ be a prime. Show that p can only be divisible by at most p prime ideals $\mathfrak{p}_i \lhd \mathcal{O}_K$ with $f_i = 1$ (ie. $\mathcal{O}_K/\mathfrak{p}_i \cong \mathbb{F}_p$).

   (*b*) (2 points) Verify that the prime 3 splits completely in $K = \mathbb{Q}(\sqrt{7}, \sqrt{10})$, so in particular there is no $\alpha \in K$ with $\mathcal{O}_K = \mathbb{Z}[\alpha]$.

   (*c*) (3 points) Find an integer $0 < N \in \mathbb{Z}$ and a suitable element $\alpha \in \mathcal{O}_K$ such that $\mathcal{O}_K[1/N] = \mathbb{Z}[1/N][\alpha]$ and factorize each rational prime $p \mid N$ in $\mathcal{O}_K$.

7. (4 points) Let $\mathbb{Q} \leq K$ be a finite extension. Assume that a prime $\mathfrak{p} \lhd \mathcal{O}_K$ splits completely in both finite extensions $(K \leq)L$ and $(K \leq)L'$. Show that it also splits completely in the composite LL'.

8. (3 points) Let $\mathbb{Q} \leq K \leq L$ be finite extensions and let $K \leq L \leq F$ be the Galois closure of the extension L/K. Show that a prime $\mathfrak{p} \lhd \mathcal{O}_K$ splits completely in L if and only if it splits completely in F.

9. (3 points) Show that the extension $\mathbb{Q}(\sqrt{-1}, \sqrt{-5})/\mathbb{Q}(\sqrt{-5})$ is unramified everywhere.