Algebraic Number Theory

Problem Sheet 4

to be handed in on 18th October 2018.

1. (2 points) Let R and R' be integral domains and $0 \notin S \subset R$ be a multiplicatively closed subset. Assume that $\varphi\colon R \to R'$ is a ring homomorphism sending S to $\varphi(S) \subseteq R'^{\times}$. Show that φ extends uniquely to a ring homomorphism $\tilde{\varphi}\colon RS^{-1} \to R'$.

2. (3 points) Let R be an integral domain and $0 \notin S \subset R$ be a multiplicatively closed subset. Show that RS^{-1} is flat as an R-module, ie. $RS^{-1} \otimes_R \cdot$ is exact.

3. (3 points) Let R be an integral domain and $f\colon M \to N$ an R-homomorphism of R-modules M, N. Prove that the following are equivalent:

   (i) f is injective (resp. surjective);

   (ii) $f_{\mathfrak{p}}\colon M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is injective (resp. surjective) for all prime ideals $\mathfrak{p} \lhd R$;

   (iii) $f_{\mathfrak{m}}\colon M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is injective (resp. surjective) for all maximal ideals $\mathfrak{m} \lhd R$.

   Here $M_{\mathfrak{p}} = R_{\mathfrak{p}} \otimes_R M$, $M_{\mathfrak{m}} = R_{\mathfrak{m}} \otimes_R M$, $N_{\mathfrak{p}} = R_{\mathfrak{p}} \otimes_R N$, and $N_{\mathfrak{m}} = R_{\mathfrak{m}} \otimes_R N$ are the localizations.

4. (3 points) Let R be an integral domain and $0 \notin S \subset R$ be a multiplicatively closed subset. Assume that RS^{-1} is integral over R (ie. each element is integral). Show $RS^{-1} = R$.

5. (3 points) (Nakayama's Lemma) Let R be a local ring with maximal ideal $\mathfrak{m} \lhd R$ and M be a finitely generated R-module with a given submodule $N \leq M$ such that $M = N + \mathfrak{m}M$. Show $M = N$.

6. (3 points) Let K be a field and $v\colon K^{\times} \to \mathbb{Z}$ be a surjective group homomorphism. Put $v(0) = \infty$ and assume $v(x + y) \geq \min(v(x), v(y))$. Show that $\mathcal{O} := \{\alpha \in K \mid v(\alpha) \geq 0\}$ is a DVR.

7. (3 points) Show that a noetherian integral domain is a DVR if and only if it is integrally closed and it has a unique nonzero prime ideal.

8. (3 points) Let $\mathcal{O}$ be a Dedekind domain and k be a positive integer. Let $\mathfrak{p}_i \lhd \mathcal{O}$ be a prime ideal, $x_i \in K$ (K being the field of fractions), and $n_i \in \mathbb{Z}$ for $1 \leq i \leq k$. Verify that there exists an $x \in K$ such that $v_{\mathfrak{p}_i}(x - x_i) \geq n_i$ ($i = 1, \ldots, k$) and $v_{\mathfrak{p}}(x) \geq 0$ for all primes $\mathfrak{p} \neq \mathfrak{p}_i$.

9. Let $K/\mathbb{Q}$ be a finite extension with ring of integers $\mathcal{O}_K$. Further let S be a finite set of nonzero prime ideals in $\mathcal{O}_K$ and $X = \mathrm{Spec}(\mathcal{O}_K) \setminus S$.

   (a) (exactness in the middle: 2 points each, exactness at each end 1 point) Verify that the sequence

   $$1 \to \mathcal{O}^\times \to \mathcal{O}(X)^\times \to \bigoplus_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O})\setminus X} K^\times/\mathcal{O}_\mathfrak{p}^\times \to Cl(\mathcal{O}) \to Cl(\mathcal{O}(X)) \to 1\ .$$

   is exact (part of the exercise to construct the maps). Further we have $K^\times/\mathcal{O}_\mathfrak{p}^\times \cong \mathbb{Z}$ for all $0 \neq \mathfrak{p} \in \mathrm{Spec}(\mathcal{O})$ primes (1 point).

   (b) (2 points) Show that $Cl(\mathcal{O}_K(X))$ is finite. Moreover, we have $\mathcal{O}_K(X)^\times \cong \mu(K) \times \mathbb{Z}^{|S|+r+s-1}$ where r (resp. s) is the number of real (resp. pairs of complex) embeddings.

10. (3 points) (Newton polygon) Let $\mathcal{O}$ be a DVR with field of fractions K and valuation $v\colon K^\times \to \mathbb{Z}$. The Newton polygon of a polynomial $f(x) = \sum_{i=0}^n a_i x^i \in K[x]$ of degree n is defined as the lower convex hull of the points $(-i, v(a_i))$ on the plane (ie. it is a convex broken line connecting $(-n, v(a_n))$ and $(0, v(a_0))$ with vertices some of $(-i, v(a_i))$ such that none of these is below the broken line). Further, put $S(f)$ for the multiset of slopes of the Newton polygon: ie. $S(f)$ has exactly n elements in which the multiplicity of a rational number s is the length of the projection of the segment of the Newton polygon with slope s to the x-axis. Show $S(fg) = S(f) \cup S(g)$. In particular, if the Newton polygon of a polynomial $f(x) \in K[x]$ consists of a single segment not containing any lattice point apart from the endpoints then f is irreducible.