Algebraic Number Theory

Problem sheet 3

to be handed in on 11th October 2018

1. (3 points) Verify the Chinese Remainder Theorem for Dedekind domains, ie. if $A \lhd \mathcal{O}$ is an ideal with decomposition $A = P_1^{\nu_1} \dots P_t^{\nu_t}$ into the product of prime ideals then we have
$$\mathcal{O}/A \cong \bigoplus_{i=1}^{t} \mathcal{O}/P_i^{\nu_i} \ .$$

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2. (3 points) Show that a lattice $\Gamma \subset V$ is full if and only if V/Γ is compact.

3. (2 points) Let $L_i(x_1, \dots, x_n) = \sum_{j=1}^{n} a_{ij} x_j$ $(i = 1, \dots, n)$ be real homogeneous polynomials of degree 1 such that $\det((a_{ij}))_{ij} \neq 0$ and let $c_1, \dots, c_n$ positive real numbers with $c_1 \dots c_n > |\det((a_{ij}))_{ij}|$. Prove that there exist integers $m_1, \dots, m_n \in \mathbb{Z}$ (not all zero) such that $|L_i(m_1, \dots, m_n)| < c_i$ $(i = 1, \dots, n)$.

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4. (2+2 points) Verify that the map
$$\begin{array}{rcl} \mathbb{C} \otimes_{\mathbb{Q}} K & \to & K_{\mathbb{C}} \\ z \otimes \alpha & \mapsto & z \cdot (j\alpha) \end{array}$$
is an isomorphism of rings. Show, moreover, that its restriction to $\mathbb{R} \otimes_{\mathbb{Q}} K$ induces an isomorphism between $K \otimes_{\mathbb{Q}} \mathbb{R}$ and $K_{\mathbb{R}}$.

5. (1 point each) Show that the rings of integers of quadratic extensions of $\mathbb{Q}$ with discriminant $5, 8, 11, -3, -4, -7, -8$, and -11 are all unique factorisation domains.

6. (*a*) (4 points) Let $X_t := \{z \in K_{\mathbb{R}} \mid \sum_\tau |z_\tau| < t\}$ for any positive real number t. Show that $\mathrm{vol}(X_t) = 2^r \pi^s \frac{t^n}{n!}$.

   (*b*) (4 points) Prove that $|d_K|^{1/2} \geq \frac{n^n}{n!} \left(\frac{\pi}{4}\right)^{n/2}$ for any finite extension $K/\mathbb{Q}$ of degree n. (Hint: Choose t so that $\mathrm{vol}(X_t) > 2^n \mathrm{vol}(\Gamma)$ where $\Gamma = j(\mathcal{O}_K) \subset K_{\mathbb{R}}$. Apply the inequality between arithmetic and geometric means on the numbers $|\tau(\alpha)|$ where $0 \neq \alpha \in \mathcal{O}_K$ is the element provided by Minkowski's Lattice Point Theorem satisfying $j(\alpha) \in X_t$. Finally, note that $1 \leq |N_{K/\mathbb{Q}}(\alpha)|$.)

   (*c*) (1 points) Show that $|d_K|$ tends to infinity as long as so does $|K : \mathbb{Q}|$. Moreover, we have $d_K > 1$ for any nontrivial extension $K/\mathbb{Q}$.

7. (*a*) (3 points) Let $A \lhd \mathcal{O}_K$ be an ideal whose class in the ideal class group has order m (ie. $A^m = (\alpha)$ is principal). Prove that $A\mathcal{O}_L = (\beta)$ where $L = K(\beta)$ with $\beta^m = \alpha$.

(*b*) (1 point) Show that for any finite extension $K/\mathbb{Q}$ there exists a finite extension L/K such that each ideal of $\mathcal{O}_K$ becomes principal over L.

(*c*) (2 points) Show that the ring Ω of all algebraic integers in $\mathbb{C}$ is a *Bézout domain*, ie. each finitely generated ideal is principal (but not noetherian).

8. (*a*) (3 points) Show that any finitely generated torsion-free module over a Bézout domain is free.

   (*b*) (2 points) Verify that $\bigcup_{n=1}^{\infty} \mathbb{C}[[x^{1/n}]]$ is a Bézout domain.

9. The goal in this exercise is to compute the rank of $\mathcal{O}_K^\times$ as an abelian group using a multiplicative version of Minkowski's theory.

   (*a*) (3 points) Let $K_\mathbb{C}^\times = \prod_\tau \mathbb{C}^\times$ be the multiplicative group of $K_\mathbb{C}$ and $N\colon K_\mathbb{C}^\times \to \mathbb{C}^\times$ be the group homomorphism sending an element to the product of coordinates. Further, define the map $l := \log|\cdot|\colon K_\mathbb{C}^\times \to \prod_\tau \mathbb{R}$ coordinate-wise. Show that the kernel of $l \circ j\colon \mathcal{O}_K^\times \to \prod_\tau \mathbb{R}$ is the torsion subgroup of $\mathcal{O}_K^\times$, ie. the group $\mu(K)$ of roots of unity in K.

   (*b*) (2 points) Verify that $l \circ j(\alpha)$ is fixed by the complex conjugation on $\prod_\tau \mathbb{R}$ (that only permutes the maps τ) for any $\alpha \in \mathcal{O}_K^\times$. Show, moreover, that the sum of coordinates of $l \circ j(\alpha)$ equals 0.

   (*c*) (5 points) Prove that $l \circ j(\mathcal{O}_K^\times)$ is a full lattice in the subspace H where we put

$$H = \{x_\tau \in \prod_\tau \mathbb{R} \mid \sum_\tau x_\tau = 0 \text{ and } x_{\sigma_k} = x_{\overline{\sigma_k}} \ (k = 1, \dots, s)\} \ .$$

   In particular, we have $\mathcal{O}_K \cong \mu(K) \times \mathbb{Z}^{r+s-1}$ as an abelian group.