Algebraic Number Theory

Problem sheet 2

to be handed in on 27th September 2018

1. (2 pont) Find an integral basis in the ring of integers in $\mathbb{Q}(\sqrt{d})$ for $d \in \mathbb{Z}$ squarefree.

2. (2 pont) Let $\mathcal{O}$ be the ring of integers in $\mathbb{Q}(\sqrt{d})$ ($d \in \mathbb{Z}$ squarefree) and $p \nmid 2d$ be a prime number. Show that $p\mathcal{O}$ is a prime ideal if and only if the Legendre symbol $\left(\frac{d}{p}\right)$ equals -1.

3. (3 pont) Prove that the ring $\mathbb{C}[x, y]/(y^2 - x^3)$ is *not* integrally closed.

4. (4 pont) Let $K/\mathbb{Q}$ be a finite extension (ie. K is a *number field*) and d_K be the discriminant of K. Prove that $d_K \equiv 0, 1 \pmod 4$. (Hint: Consider the permanent of the matrix $((\sigma_i \alpha_j))$ (the permanent of a matrix is similar to the determinant but we take all the terms in the formula with positive sign)? Use the identity $(a - b)^2 = (a + b)^2 - 4ab$.)

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5. (2 pont) Write $33 + 11\sqrt{-7}$ as a product of irreducible elements in the ring of integers of $\mathbb{Q}(\sqrt{-7})$.

6. (3 pont) Show that a Dedekind domain with finitely many prime ideals is in fact a principal ideal domain.

7. (3+3 pont)

   (i) Let $\mathcal{O}$ be a Dedekind domain and $I \lhd \mathcal{O}$ be an ideal. Show that each ideal in $\mathcal{O}/I$ is principal (ie. generated by one element).

   (ii) Show that each ideal in a Dedekind domain is generated by at most two elements.

8. (5 pont) Let $\mathcal{O}$ be an integral domain such that any ideal in $\mathcal{O}$ can be uniquely written as a product of prime ideals. Show that $\mathcal{O}$ is a Dedekind domain (ie. integrally closed, noetherian, and 1-dimensional).

9. (4 pont) Show that any ideal in a Dedekind domain is a projective module.

10. (3+3 pont) Let $f(x, y) \in \mathbb{C}[x, y]$ be an irreducible nonsingular polynomial (ie. $(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (1)$ as an ideal in $\mathbb{C}[x, y]$). Show that $\mathbb{C}[x, y]/(f(x, y))$ is a Dedekind domain. What will be the class group if the zeros of f form an elliptic curve, ie. $f(x, y) = y^2 - x^3 - ax - b$ where $x^3 + ax + b$ has no multiple roots? (Compare to Problem 3 where the curve is *singular*.)