Algebraic Nuber Theory

Problem Sheet 10

to be handed in until 6th December 2018

1. (2 points) Let $K/\mathbb{Q}_p$ be a finite extension. Verify that the formal series $\log(1+x) = x - \frac{x^2}{2} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \ldots$ converges on the maximal ideal of the valuation ring.
2. (3 points) Determine the radius of (p-adic) convergence of the series $\exp(x) = 1 + x + \cdots + \frac{x^n}{n!} + \ldots$ in finite extensions $K/\mathbb{Q}_p$.
3. (3 points) Let $K/\mathbb{Q}_p$ be a finite extension with absolute ramification index e and maximal ideal $\mathfrak{p}$. Verify that $\exp\colon \mathfrak{p}^n \to U^{(n)} = 1 + \mathfrak{p}^n$ and $\log\colon U^{(n)} \to \mathfrak{p}^n$ are inverse isomorphisms between the additive group $\mathfrak{p}^n$ and the multiplicative group $U^{(n)}$ if $n > \frac{e}{p-1}$. In particular, $U^{(n)}$ is torsion free if $n > \frac{e}{p-1}$.
4. (3 points) Let $|K : \mathbb{Q}_p| = d$, $\pi \in K$ be the prime element and put $q = \mathcal{O}_K/(\pi)$. Show that we have $K^\times \cong \pi^{\mathbb{Z}} \oplus Z_{q-1} \oplus Z_{p^a} \oplus \mathbb{Z}_p^d$ for some integer $a \geq 0$.
5. (2 points) Let $z \in \mathbb{Z}_p$ be fixed. Show that the binomial series $(1+x)^z = \sum_{n=0}^{\infty} \binom{z}{n} x^n$ converges for $v_p(x) > \frac{1}{p-1}$. Moreover, in this case we have $(1+x)^z = \exp(z \log(1+x))$.
6. (3 points) Let $K/\mathbb{Q}_p$ be finite. Prove that any finite index subgroup of $K^\times$ is open (hence closed).
7. (3 points) Let $K/\mathbb{Q}_p$ be finite, $\pi \in K$ be a prime, and v_π the normalized valuation (so that $v_\pi(\pi) = 1$). Show that $\mathcal{O}_K$ is compact, in particular K is a locally compact abelian group. So there exists a translation-invariant Haar measure dx which is unique if we assume $\int_{\mathcal{O}_K} dx = 1$. Verify $v_\pi(a) = \int_{a\mathcal{O}_K} dx$. Further show that $\frac{dx}{|x|_\pi}$ is a translation-invariant Haar measure on the group $K^\times$.
8. (3 points) Let $K/\mathbb{Q}_p$ be finite. Show that the slopes of the Newton polygon of a polynomial $f(x) \in K[x]$ are exactly $\{v_\pi(\alpha_i) \mid 1 \leq i \leq n\}$ (with muliplicities) where $\pi \in K$ is a prime and f decomposes as $f(x) = c\prod_{i=1}^{n}(x - \alpha_i)$ over $\overline{K}$ (the Newton-polygon is also constructed using the π-adic valuation).
9. (3 points) Let $K/\mathbb{Q}_p$ be finite and $f(x) \in K[x]$ be an irreducible polynomial. Show that the Newton polygon of f consists of a single segment. Give an example that it may contain a lattice point apart from the two endpoints.

---

10. (3 points) Let R be a commutative ring with identity. A (one-parameter) commutative formal group law is a formal power series $F(X,Y) \in R[[X,Y]]$ in two variables satisfying

(*i*) $F(X,Y) = X + Y$+terms of higher degree;

(*ii*) $F(X, F(Y,Z)) = F(F(X,Y),Z)$ (associative);

(*iii*) there exists a power series $\iota_F(X) \in XR[[X]]$ such that $F(X, \iota_F(X)) = 0$ (inverse);

(*iv*) $F(X,Y) = F(Y,X)$ (commutative).

Now assume $R = \mathcal{O}_K$ for some complete non-archimedean field K with maximal ideal $\mathcal{M}_K$. Show that $\mathcal{M}_K$ is an abelian group with respect to the operation $a +_F b := F(a,b)$. Further verify that power series $F(X,Y) = X + Y$ and $F(X,Y) = X + Y + XY$ are commutative formal group laws. Describe $(\mathcal{M}_K, +_F)$ in this case (upto isomorphism)?

11. (3 points) A homomorphism from the formal group F to G is a power series $h(T) \in TR[[T]]$ such that $h(F(X,Y)) = G(h(X), h(Y))$. Show that in case $F = G$ the set $\mathrm{End}(F) := \mathrm{Hom}(F,F)$ is a ring with respect to the addition $+_F$ and multiplication given by composition. Further verify that $h_n(T) = (T+1)^n - 1$ is an endomorphism of the formal group law $F(X,Y) = X + Y + XY$ for all $n \geq 1$.

12. (4 points) From now on assume $R = \mathcal{O}_K$ where $K/\mathbb{Q}_p$ is a finite extension and $\pi \in \mathcal{O}_K$ is a prime. Let $\mathcal{F}_\pi$ be the set of power series $f(X) \in \mathcal{O}_K[[X]]$ such that $f(X) = \pi X$+terms of higher degree and $f(X) \equiv X^q \pmod{\pi}$ where $q = p^f$ is the cardinality of the residue field $k = \mathcal{O}_K/(\pi)$. Let $f, g \in \mathcal{F}_\pi$ and $\phi_1(X_1, \dots, X_n) \in \mathcal{O}_K[X_1, \dots, X_n]$ be a homogeneous polynomial of degree 1. Verify that there exists a unique power series $\phi(X_1, \dots, X_n) \in \mathcal{O}_K[[X_1, \dots, X_n]]$ such that $\phi(X_1, \dots, X_n) = \phi_1(X_1, \dots, X_n)$+terms of higher degree and $f(\phi(X_1, \dots, X_n)) = \phi(g(X_1), \dots, g(X_n))$.

13. (3 points) Prove that for all $f \in \mathcal{F}_\pi$ there exists a unique formal group law $F_f(X,Y) \in \mathcal{O}_K[[X,Y]]$ such that f is an endomorphism of F_f. (These are called Lubin–Tate formal group laws.)

14. (3 points) Let $a \in \mathcal{O}_K$ and $f, g \in \mathcal{F}_\pi$. Show that there exists a unique power series $[a]_{g,f}(T) \in T\mathcal{O}_K[[T]]$ such that $[a]_{g,f}(T) = aT$+terms of higher degree and $[a]_{g,f} \circ f = g \circ [a]_{g,f}$. Moreover, $[a]_{g,f}$ is a homomorphism from F_f to F_g. In particular F_f and F_g are isomorphic.

15. (2 points) Verify that the map $\mathcal{O}_K \to \mathrm{End}(F_f)$, $a \mapsto [a]_f := [a]_{f,f}$ is an injective ring homomorphism with $[\pi]_f = f$. (This makes F_f a *formal $\mathcal{O}_K$-module.*)

16. (3 points) Let $f \in \mathcal{F}_\pi$ be arbitrary (for the sake of simplicity we may take the polynomial $f(T) = \pi T + T^q$ by Problem 14) and put Λ_n for the set of roots of the polynomial $f^{(n)} = \underbrace{f \circ \cdots \circ f}_{n}$ in an algebraic closure of $\mathbb{Q}_p$. Verify that Λ_n is an $\mathcal{O}_K$-module of cardinality q^n with respect to the addition $+_{F_f}$ and multiplication $a \cdot_{F_f} \lambda := [a]_f(\lambda)$ ($a \in \mathcal{O}_K$, $\lambda \in \Lambda_n$). Further show $\Lambda_n \cong \mathcal{O}_K/(\pi^n)$ as $\mathcal{O}_K$-modules.

17. (4 points) Let $K_{\pi,n} := K(\Lambda_n)$ be the splitting field of the polynomial $f^{(n)}$ over K. Verify $\mathrm{Gal}(K_{\pi,n}/K) \cong (\mathcal{O}_K/(\pi^n))^\times$ and that $K_{\pi,n}/K$ is totally ramified. Further show that $K_{\pi,n}$ does not depend on the choice of $f \in \mathcal{F}_\pi$ and π is the norm of a suitable element in $K_{\pi,n}$.