$p$-adic Galois representations

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Talk at Heidelberg

6th June 2019
Riemann’s zeta function

$L$-functions are attached to various objects in arithmetic geometry.

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} \quad (\Re(s) > 1)
\]

Encoded arithmetic information:
- Distribution of primes: zeros in the critical strip $0 < \Re(s) < 1$
- Arithmetic of cyclotomic fields $\mathbb{Q}(\mu_p)$: special values $\zeta(-1), \zeta(-3), ..., \zeta(2-p)$ $\mapsto p$-adic \(\zeta\)-function by $p$-adic interpolation

Need analytic continuation and functional equation!
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$L(E, s) := \prod_{p \text{ prime}} \frac{1}{P_{E,p}(p^{-s})}$

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$P_{E,p}(T) = 1 - a_p T + p T^2$ if $E$ has good reduction at $p$

where

$\#E(\mathbb{F}_p) = P_{E,p}(1) = 1 - a_p + p$. 

Encoded arithmetic information:

Number of mod $p$ points $E(\mathbb{F}_p)$

Conjecturally: number of rational points:

Conjecture of Birch and Swinnerton-Dyer (1960s) – weak form

$L(E, 1) = 0$ if and only if $\#E(\mathbb{Q}) = \infty$.

Analytic continuation in this case: Taniyama–Shimura–Weil conjecture (proven by Wiles and Taylor (1993)).
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Let $E$ be an elliptic curve defined over $\mathbb{Q} \rightsquigarrow$ $L$-function

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Varieties $\rightsquigarrow$ Galois representations

Let $X$ be a smooth projective variety defined over $\mathbb{Q}$ and put $G_{\mathbb{Q}} := \text{Gal} (\overline{\mathbb{Q}}/\mathbb{Q})$. For any prime $\ell$ and integer $i \geq 0$ we have an action of $G_{\mathbb{Q}}$ on the $\ell$-adic cohomology group

$$H^i_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) := \left( \lim_{\leftarrow r} H^i_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell^r \mathbb{Z}) \right) [\ell^{-1}] .$$

Reason for finite coefficients:

$H^i_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell^r \mathbb{Z}) \cong H^i_{sing}(X(\mathbb{C}), \mathbb{Z}/\ell^r \mathbb{Z})$. Need to pass to characteristic 0 in order to define $L$-functions $\rightsquigarrow \ell$-adic representations!
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- $X = \{\ast\}, i = 0 \leadsto$ trivial Galois representation.
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In the above examples:

- $X = \{\ast\}, i = 0 \rightsquigarrow$ trivial Galois representation.
- $X = E, i = 1 \rightsquigarrow H^1_{\text{et}}(E_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell^r \mathbb{Z}) \cong E[\ell^r](1).$
Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ for any prime $p$ (and also $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$). This defines an embedding $G_{\overline{\mathbb{Q}}_p} \hookrightarrow G_{\overline{\mathbb{Q}}}$.
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$$1 \rightarrow I_p \rightarrow G_{\mathbb{Q}_p} \rightarrow G_{\mathbb{F}_p} \rightarrow 1$$

where $G_{\mathbb{F}_p} \cong \hat{\mathbb{Z}}$ is topologically generated by the (arithmetic) Frobenius automorphism $\text{Frob}_p : x \mapsto x^p$. 
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where $G_{\mathbb{F}_p} \cong \hat{\mathbb{Z}}$ is topologically generated by the (arithmetic) Frobenius automorphism $\text{Frob}_p : x \mapsto x^p$. Now if

$$\rho : G_{\mathbb{Q}} \to \text{GL}(V)$$

is a global Galois-representation on a finite dimensional vectorspace $V$ over a field $K$ of characteristic 0 (embedded into $\mathbb{C}$) then we defined the local polynomial at $p$ as the characteristic polynomial

$$P_{\rho,p}(T) := \det(id - T \text{Frob}_p | V^{I_p}) \in K[T].$$
Galois representations $\rightsquigarrow L$-functions

The $L$-function attached to the Galois representation $\rho$ is defined as

$$L(\rho, s) := \prod_{p \text{ prime}} \frac{1}{P_{\rho,p}(p^{-s})} \quad (\text{Re}(s) \gg 0).$$

In case of $X = \{\ast\}, i = 0$ this specializes to Riemann $\zeta$ and in case $X = E, i = 1$ to the $L$-function of the elliptic curve as above.
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Fundamental open questions in the theory:

- Analytic continuation and functional equation $\leadsto$ modularity
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Fundamental open questions in the theory:
- Analytic continuation and functional equation $\leadsto$ modularity
- Which Galois representations arise from geometry, ie. as a subquotient of the étale cohomology of a smooth projective variety?

The above 2 questions are closely related.
Fontaine–Mazur conjecture (1995)

An irred. \( \ell \)-adic Galois representation \( \rho : G_{\mathbb{Q}} \rightarrow \text{GL}_n(\mathbb{Q}_\ell) \) comes from geometry if and only if the following two conditions hold:

(i) \( \rho \) is unramified (i.e. \( \rho(I_p) = \{1\} \)) at all but finitely many primes \( p \).

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The “only if” part of the above conjecture is known: $i)$ by Grothendieck (note that in the case of elliptic curves those primes ramify at which the curve has bad reduction: criterion of Néron–Ogg–Shafarevich—in particular, there are finitely many). Assertion $ii)$ (“$p$-adic de Rham comparison isomorphism”) was first proven by Faltings and by Tsuji and reproven recently by Beilinson (survey: Szamuely–Z) and by Scholze.
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Assertion (ii) (“\(p\)-adic de Rham comparison isomorphism”) was first proven by Faltings and by Tsuji and reproven recently by Beilinson (survey: Szamuely–Z) and by Scholze. We need to better understand the case \(\ell = p\)!
Let \( X \) be a smooth projective variety over \( \mathbb{C} \). Classical Poincaré lemma

\[
H^n_{\text{sing}}(X(\mathbb{C}), \mathbb{C}) = H^n_{\text{dR}}(X^{an}, \mathbb{C})
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where the right hand side is computed by the Hodge–to–de Rham spectral sequence

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E_1^{p,q} := H^q(X^{an}, \Omega^p_{X^{an}}) \Rightarrow H^{p+q}_{\text{dR}}(X^{an}, \mathbb{C})
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where \( \Omega^p_{X^{an}} \) stands for the sheaf of holomorphic \( p \)-forms on the analytic manifold \( X^{an} \).
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Can we generalize this to other ground fields $K$?
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- Étale cohomology can be regarded as the analogue of singular cohomology: they agree if $K = \mathbb{C}$ and the coefficients are finite (or, after taking the limit, $p$-adic).
- In case of algebraic de Rham cohomology coefficients lie in $K$!
So we take $K = \mathbb{Q}_p$. Associated to the algebraic de Rham complex

$$\Omega^\bullet_X : \mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2_X \to \cdots$$

of sheaves (in the Zariski topology) of Kähler-differentials there is a Hodge–to–de Rham spectral sequence

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For a $p$-adic Poincaré lemma to hold, one has to pass to a big field $\mathcal{B}_{dR}$ (which is a discretely valued field with residue field $\mathbb{C}_p = \hat{\mathbb{Q}}_p$ admitting an action of $G_{\mathbb{Q}_p}$) so one has an isomorphism (Faltings)

$$H^i_{dR}(X/\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathcal{B}_{dR} \xrightarrow{\sim} H^i_{et}(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathcal{B}_{dR}$$

compatible with the filtration and the Galois action on both sides.
Taking $G_{\mathbb{Q}_p}$-invariants of the isomorphism above one obtains

$$H_{dR}^i(X/\mathbb{Q}_p) \cong \left( H^i_{\text{et}}(\overline{X}_{\mathbb{Q}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{dR}\right)^{G_{\mathbb{Q}_p}}$$

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using the fact $B_{dR}^{G_{Q_p}} = Q_p$. By GAGA the two sides have the same dimension therefore we define a local $p$-adic Galois-representation $V$ to be de Rham if we have $\dim_{Q_p} D_{dR}(V) = \dim_{Q_p} V$ where

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Problem: We cannot recover $V$ from $D_{dR}(V)$! (even if $V$ is de Rham)
Galois representation in characteristic $p$

Let $E$ be a perfect field of characteristic $p$ and $V$ be a finite dimensional representation of $G_E := \text{Gal}(\overline{E}/E)$ over $\mathbb{F}_p$. By Hilbert’s Theorem 90 we can trivialize $V$ over $\overline{E}$, i.e.

$$ \overline{E} \otimes_{\mathbb{F}_p} V \cong \overline{E}^{\dim_{\mathbb{F}_p} V} \cong \overline{E} \otimes_E \left( \overline{E} \otimes_{\mathbb{F}_p} V \right)^{G_E} $$

as $G_E$-modules. In particular, $D(V) := \left( \overline{E} \otimes_{\mathbb{F}_p} V \right)^{G_E}$ has dimension $\dim_{\mathbb{F}_p} V$ over $E$. 
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Key extra structure: in characteristic $p$ the Frobenius $\text{Frob}_p : \bar{E} \to \bar{E}$ has fixed field $\mathbb{F}_p$.

Put $\varphi := \text{Frob}_p \otimes id_V : \bar{E} \otimes_{\mathbb{F}_p} V \to \bar{E} \otimes_{\mathbb{F}_p} V$ so we have $V = (\bar{E} \otimes_{E} D(V))^{\varphi=id}$. 
How to pass from char 0 to char $p$?

Tilting equivalence of Scholze!
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- Scholze ($\sim$2012) extended the notion and made it more geometric

**Definition**

Let $K$ be a field that is complete with respect to a nonarchimedean *nondiscrete* valuation $| \cdot | : K \to \mathbb{R}_{\geq 0}$. We say that $K$ is **perfectoid** if the $p$-Frobenius map $\text{Frob}_p : \mathcal{O}_K/(p) \to \mathcal{O}_K/(p)$ is surjective.
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Examples: $\mathbb{C}_p$, $\hat{\mathbb{Q}}_p(\mu_{p^\infty})$, $\hat{\mathbb{Q}}_p(p^{1/p^\infty})$, $\mathbb{F}_p((\mathbb{T}^{1/p^\infty}))$ but not $\mathbb{Q}_p$ (valuation is discrete!).
Tilting equivalence

Let $K$ be a perfectoid field. The perfectoid field $K^b := \text{Frac}(\mathcal{O}_{K^b})$ of characteristic $p$ is called the *tilt* of $K$ where

$$
\mathcal{O}_{K^b} := \lim_{\xleftarrow{\text{Frob}_p}} \mathcal{O}_K/(p).
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Let $K$ be a perfectoid field. The perfectoid field $K^\flat := \text{Frac}(\mathcal{O}_{K^\flat})$ of characteristic $p$ is called the *tilt* of $K$ where

$$\mathcal{O}_{K^\flat} := \varprojlim \mathcal{O}_K/(p).$$

**Theorem (Tilting equivalence of Scholze)**

Let $K$ be a perfectoid field. Then the functor $\flat : L \mapsto L^\flat$ gives an equivalence of categories between perfectoid extensions of $K$ and perfectoid extensions of $K^\flat$. Moreover, if $L/K$ is finite separable then $L$ is perfectoid (baby case of almost purity).
Tilting equivalence

Let $K$ be a perfectoid field. The perfectoid field $K^b := \text{Frac}(\mathcal{O}_{K^b})$ of characteristic $p$ is called the \textit{tilt} of $K$ where

$$\mathcal{O}_{K^b} := \lim_{\overset{\leftarrow}{\text{Frob}_p}} \mathcal{O}_K/(p).$$

\textbf{Theorem (Tilting equivalence of Scholze)}

Let $K$ be a perfectoid field. Then the functor $\♭ : L \mapsto L^♭$ gives an equivalence of categories between perfectoid extensions of $K$ and perfectoid extensions of $K^♭$. Moreover, if $L/K$ is finite separable then $L$ is perfectoid (baby case of almost purity).

\textbf{Corollary}

We have $G_K \cong G_{K^♭}$ and if $K$ is the completion of a Galois extension of $\mathbb{Q}_p$ then we have $\text{Gal}(K/\mathbb{Q}_p) \hookrightarrow \text{Aut}(K^♭)$. 
Let $K$ be a perfectoid field (of char 0)

$\{\text{mod } p \text{ reps of } G_K\} \leftrightarrow \{\text{mod } p \text{ reps of } G_{K^b}\} \leftrightarrow \{\varphi\text{-modules }/K^b\}$
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By taking Witt vectors and inverting $p$ we also have

\[ \{ p\text{-adic reps of } G_K \} \leftrightarrow \{ p\text{-adic reps of } G_{K^b} \} \leftrightarrow \{ \varphi\text{-mods }/W(K^b)[p^{-1}] \} \]
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**New feature** (Scholze): There is a geometric object $\text{Spd}(\mathbb{Q}_p)$ in characteristic $p$ with étale fundamental group $G_{\mathbb{Q}_p}$: formal orbit space of $\Gamma$-action on $\text{Spa}(\mathbb{Q}_p(\mu_{p^\infty})^b)$ in the category of diamonds.
Imperfect \((\varphi, \Gamma)\)-modules

We have \(\overline{\mathbb{Q}_p(\mu_{p\infty})}^b = \overline{\mathbb{F}_p((T^{1/p\infty})^b)}\)—one could, for most purposes, work with \((\varphi, \Gamma)\)-modules over these. But e.g. for the \(p\)-adic Langlands programme one needs *imperfect* ground fields.
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We have \(\widehat{\mathbb{Q}_p}(\mu_{p^\infty})^b = \mathbb{F}_p((\mathcal{T}^{1/p^\infty}))\)—one could, for most purposes, work with \((\varphi, \Gamma)\)-modules over these. But e.g. for the \(p\)-adic Langlands programme one needs *imperfect* ground fields.

Observation: we have \(G_{\mathbb{F}_p((\mathcal{T})')} \cong G_{\mathbb{F}_p((\mathcal{T}^{1/p^\infty})')} \mapsto \{\text{mod } p \text{ reps of } G_{\mathbb{Q}_p}\} \leftrightarrow \{((\varphi, \Gamma)\text{-modules} / \mathbb{F}_p((\mathcal{T}))\} \)}}
Imperfect \((\varphi, \Gamma)\)-modules

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Observation: we have
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G_{\mathbb{F}_p((T))} \cong G_{\mathbb{F}_p((\overline{T^{1/p^{\infty}}})^b)} \mapsto
\]
\[
\{ \text{mod } p \text{ reps of } G_{\mathbb{Q}_p} \} \leftrightarrow \{(\varphi, \Gamma)\text{-modules } / \mathbb{F}_p((T))\}
\]
and
\[
\{ \text{p-adic reps of } G_{\mathbb{Q}_p} \} \leftrightarrow \{ \text{étale } (\varphi, \Gamma)\text{-modules } / \mathcal{E} \}
\]
where we put \(\mathcal{E} := \mathcal{O}_\mathcal{E}[p^{-1}]\) and \(\mathcal{O}_\mathcal{E} := \lim_n \mathbb{Z}/(p^n)((T))\). Étale means: \(\text{id} \otimes \varphi: \mathcal{E} \otimes_{\mathcal{E}, \varphi} D \to D\) is bijective (note: \(\text{Frob}_p: \mathbb{F}_p((T)) \to \mathbb{F}_p((T))\) is no longer bijective!) This is Fontaine’s equivalence of categories (1990).
Let $V$ be a $p$-adic representation of $G_{\mathbb{Q}_p}$ and put $D(V)$ for the corresponding $(\varphi, \Gamma)$-module over $E$. In order to recover $D_{dR}(V) = (\mathcal{B}_{dR} \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}$ from $D(V)$ one first has to pass to coefficient rings converging $p$-adically at least somewhere.
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\[
\mathcal{R}^{(r,1)} := \{ f(T) = \sum_{i=-\infty}^{\infty} a_i T^i \mid a_i \in \mathbb{Q}_p, \text{ f converges if } r < |T|_p < 1 \}
\]

\[
\mathcal{R} := \bigcup_{0<r<1} \mathcal{R}^{(r,1)} \quad \mathcal{E}^\dagger := \{ f \in \mathcal{R} \mid \limsup_{|T|_p \to 1} |f(T)|_p < \infty \}
\]

Note: $\mathcal{E}^\dagger$ embeds into $\mathcal{E}$ (but $\mathcal{R}$ does not).
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$$R^{(r,1)} := \left\{ f(T) = \sum_{i=-\infty}^{\infty} a_i T^i \mid a_i \in \mathbb{Q}_p, \text{ } f \text{ converges if } r < |T|_p < 1 \right\}$$

$$R := \bigcup_{0 < r < 1} R^{(r,1)} \quad \quad \mathcal{E}^\dagger := \left\{ f \in R \mid \limsup_{|T|_p \to 1} |f(T)|_p < \infty \right\}$$

Note: $\mathcal{E}^\dagger$ embeds into $\mathcal{E}$ (but $R$ does not).

**Theorem (Cherbonnier–Colmez: overconergence)**

$D(V)$ descends to an étale $(\varphi, \Gamma)$-module $D^\dagger(V)$ over $\mathcal{E}^\dagger$. 
Theorem (Berger)

Put \( D^{\text{rig}}(V) := \mathcal{R} \otimes_{\mathcal{E}^\dagger} D^\dagger(V) \) and
\[
t := \log(1 + T) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{T^k}{k} \in \mathcal{R}.
\]
Then there exists a \( p \)-adic differential equation (\( \mathbb{Q}_p(\mu_p)\)[[t]]-module with \( \Gamma \)-action) \( D^{\text{dif}}(V) \) associated to \( D^{\text{rig}}(V) \) such that we have
\[
D_{dR}(V) = D^{\text{dif}}(V)^\Gamma.
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Theorem (Berger)

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$t := \log(1 + T) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{T^k}{k} \in \mathcal{R}$. Then there exists a $p$-adic differential equation ($\mathbb{Q}_p(\mu_{p^\infty})[[t]]$-module with $\Gamma$-action) $D^{\text{dif}}(V)$ associated to $D^{\text{rig}}(V)$ such that we have

$$D_{dR}(V) = D^{\text{dif}}(V)^\Gamma.$$

Most applications use: for $? = \text{rig}, \uparrow, \text{or empty}$ Herr’s complex below computes Galois cohomology:

$$0 \rightarrow D?(V)^{(\varphi - \text{id}, \gamma - \text{id})} \rightarrow D?(V) \oplus D?(V)^{(\text{id} - \gamma, \varphi - \text{id})} \rightarrow D?(V) \rightarrow 0.$$
Motivation: generalize Colmez’ functors

Main observation of Colmez when constructing a $p$-adic Langlands correspondence for $\text{GL}_2(\mathbb{Q}_p)$:

$$1 + T \leftrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \varphi \leftrightarrow \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \quad \Gamma \leftrightarrow \begin{pmatrix} \mathbb{Z}_p^\times & 0 \\ 0 & 1 \end{pmatrix}$$
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$\sim$ functor from smooth (ie. stabilizers are open) mod $p^n$ representations $\mapsto$ mod $p^n$ étale $(\varphi, \Gamma)$-modules $\mapsto$ mod $p^n$ local Galois representations.
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If there is a generalization to groups of higher rank (e.g. $GL_n(\mathbb{Q}_p)$ with $n > 2$) it is natural to expect that “multivariable” objects come into picture.
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If there is a generalization to groups of higher rank (e.g. $\GL_n(\mathbb{Q}_p)$ with $n > 2$) it is natural to expect that “multivariable” objects come into picture. Hint (Breuil–Herzig–Schraen): A generalized Colmez functor applied to the automorphic $\GL_n(\mathbb{Q}_p)$-representation attached to a mod $p$ (global) Galois representation $\rho$ (the corresponding Hecke-isotypical component in the cohomology of a Shimura-variety) should not give $\rho$ back but $\bigotimes_{i=1}^n \wedge^i \rho$. 
Theorem (Z, Carter–Kedlaya–Z)

Let \( \Delta \) be a finite set and put \( G_{\mathbb{Q}_p,\Delta} := \prod_{\alpha \in \Delta} G_{\mathbb{Q}_p} \). There is an equivalence of categories

\[
\{\text{\( p \)-adic reps of } G_{\mathbb{Q}_p,\Delta}\} \leftrightarrow \{\text{étale } (\varphi_\Delta, \Gamma_\Delta)\text{-modules over } \mathcal{E}_\Delta\}
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where \( \varphi_\Delta = (\varphi_\alpha \mid \alpha \in \Delta) \) (one Frobenius lift for each variable), \( \Gamma_\Delta := \prod_{\alpha \in \Delta} \Gamma \), \( \mathcal{E}_\Delta := \mathcal{O}_{\mathcal{E}_\Delta}[p^{-1}] \) and
\[
\mathcal{O}_{\mathcal{E}_\Delta} := \varprojlim_n \mathbb{Z}/(p^n)[[T_\alpha \mid \alpha \in \Delta]][\prod_{\alpha \in \Delta} T_\alpha^{-1}].
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Theorem (Z)

There is a right exact functor compatible with parabolic induction and tensor products from the category of smooth mod $p^n$ representations of $GL_n(\mathbb{Q}_p)$ to the category of mod $p^n$ representations of $G_{\mathbb{Q}_p}^{n-1} \times \mathbb{Q}_p^\times$. In case of $n = 2$ this agrees with Colmez’ functor realizing $p$-adic Langlands for $GL_2(\mathbb{Q}_p)$. 

Methods of proof (Carter–Kedlaya–Z)

Recall: $\pi_1(Spd(\mathbb{Q}_p)) \cong G_{\mathbb{Q}_p}$. 
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Recall: $\pi_1(\text{Spd}(\mathbb{Q}_p)) \cong G_{\mathbb{Q}_p}$. Analogue of Künneth Theorem $\pi_1(X_1 \times X_2) \cong \pi_1(X_1) \times \pi_1(X_2)$ for pointed topological spaces in characteristic $p$ geometry?

$\text{Spec } k \times \text{Spec } k = \text{Spec } k$ but $\text{diag}: G_k \to G_k \times G_k$ is not an isomorphism...
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Let \( X_1, \ldots, X_n \) be connected schemes of finite type over \( \mathbb{F}_p \) and put \( X := X_1 \times \cdots \times X_n \). Let \( \varphi_i = 1 \times \cdots \times \varphi_{X_i} \times \cdots \times 1: X \to X \) be the \( i \)th partial Frobenius and denote by \( \text{F} \text{Et}(X/\Phi) \) the category of finite étale maps \( Y \to X \) equipped with commuting isomorphisms \( \beta_i: Y \to \varphi_i^* Y \) such that the “composite” \( \beta_n \circ \cdots \circ \beta_1 \) is the relative Frobenius \( \varphi_{Y/X} \).
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**Drinfeld’s lemma for schemes**

$\pi_1(X/\Phi) \cong \pi_1(X_1) \times \cdots \times \pi_1(X_n)$. 
Methods of proof (Carter–Kedlaya–Z) cont’d

Analogue of Drinfeld’s Lemma holds for connected, quasi-compact, quasi-separated diamonds $X_i$ (due to Scholze and Kedlaya).
Methods of proof (Carter–Kedlaya–Z) cont’d

Analogue of Drinfeld’s Lemma holds for connected, quasi-compact, quasi-separated diamonds $X_i$ (due to Scholze and Kedlaya).

Technical problem: $\text{Spd}(\mathbb{Q}_p)^n = (\text{Spa}(\mathbb{Q}_p(\mu_{p^\infty}))^b)^n / \Gamma^n$, but $(\text{Spa}(\mathbb{Q}_p(\mu_{p^\infty}))^b)^n$ is not an “affine diamond”, i.e. it is not the diamond spectrum of our (perfect) coefficient ring $R := R^+[(T_1 \cdots T_n)^{-1}]$ where

$$R^+ := \lim \left< \left( \mathbb{F}_p[[T_1^{p^{-\infty}}]] \otimes_{\mathbb{F}_p} \cdots \otimes_{\mathbb{F}_p} \mathbb{F}_p[[T_n^{p^{-\infty}}]] \right)/\langle T_1, \ldots, T_n \rangle \right>_r$$

but the rational subspace defined by

$$\{|T_1| < 1, \ldots, |T_n| < 1\}.$$
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Need: “Perfectoid Riemann Extension Theorem” + “imperfection”. Holds also for possibly distinct finite extensions $K_1, \ldots, K_n$ of $\mathbb{Q}_p$. 
Further results and possible future directions

- (Pal–Z) Generalization of Herr’s complex still computes group cohomology of $G_{\mathbb{Q}_p,\Delta}$.
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Future directions:

- Pass to the Robba ring and construct Bloch–Kato exponential maps and Perrin-Riou’s big exponential maps in this product situation $\rightsquigarrow$ prove classical $\varepsilon$-isomorphisms (etc.$?$) for $p$-adic representations of the form $V_1 \otimes_{\mathbb{Q}_p} V_2$ if it is known for both $V_1$ and $V_2$. 
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- (Pal–Z) Generalization of Herr’s complex still computes group cohomology of $G_{Q_p,\Delta}$.
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- Relate these notions to Berger’s Lubin–Tate multivariable $(\varphi, \Gamma)$-modules $\rightsquigarrow$ better structural properties of the latter
Thanks for your attention!